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THESIS

NONLINEAR DYNAMICS OF CLOSE PROXIMITY SHIP TOWING

by

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December 2002

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NONLINEAR DYNAMICS OF CLOSE PROXIMITY SHIP TOWING

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ABSTRACT

The goal of this study is to investigate the nonlinear dynamics of two ships in close proximity towing. The sway and yaw dynamics of both the leading and the trailing ships were included in the formulation. Previous studies were restricted to a linear analysis, which can accurately predict the regions of stability and instability for the system. The mechanism of loss of stability can be assessed with a systematic nonlinear analysis. The analysis is based on Taylor series expansions of the equations of motion up to third order terms. It is shown that the primary loss of stability occurs in the form of Hopf bifurcations to periodic solutions. A nonlinear stability coefficient was calculated which allows characterization of the stability properties of the resulting limit cycles. The results indicate the effects of ship separation and towing tension on motion stability.

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I. INTRODUCTION

One of the main concerns in both naval architecture and maneuvering control is the stability of the ship towing. In particular, the close proximity ship-towing problem has many unexplored areas of investigation ranging from building a mathematical model of the two towed ships, analyzing the directional stability of the combined system, local stability of the maneuvering, nonlinear dynamics of the system and elasticity effects in the towline.

The close proximity ship towing is under current study by Office of Naval Research. Traditionally, ship towing for the military ships and the others are done with long towlines. The Office of Naval Research is studying the close proximity towing for its innovative Hull Programs, such as small waterplane area twin hull (SWATH) ships and their variations (such as the SLICE hull). During the SEA LANCE project, 2000, at the Naval Postgraduate School (Total Ship Systems Engineering Program) it was shown that one of the main benefits of close proximity towing in military applications is the ability to separate a combatant ship from a main part of its payload.

In this thesis the focus of our efforts is on the nonlinear dynamics during the loss of directional stability in close proximity ship towing. Previous studies on the directional stability of ship towing were performed in 1964 by Abkowitz who developed the characteristic equation for single body towing, and by Bernitsas and others who developed the criteria for stability of Abkowitz 4^{th} order characteristic equation. The underlying assumption in those studies was; the leading ship is on a constant straight-line motion of a point mass. This is a valid approximation for long towlines while it may not capture the combined dynamics during towing in close proximity. Therefore, it is possible that the existing criteria of directional stability may be inadequate. The directional stability for the coupled system where both the maneuvering dynamics of leading and trailing ships are considered was studied by GOKCE in 2002. As a result, the complete system has now a dimensionality of 8 and the characteristic order is 8^{th} order instead of 4^{th}. With the inclusion of the dynamics for the leading ship, it is necessary to incorporate path control into the system. For the demonstration purposes, we include a

rudder control law for the leading ship only. This is done in terms of full state feedback in terms of the sway and yaw velocities, heading and lateral deviation of the leading ship from its desired track. Control law responsiveness is quantified via its time constant, small values of the time constant designate a responsive controller while larger values designate a more sluggish design.

In this study as a leading ship we will use the SLICE vessel, which is 105 ft, and 180 tons, and as a trailing ship the SWATH ship Kaimalino which is 89 ft, 47 ft beam, 10-12 knots speed and 217 tons. Typical speeds for the combined vessels in the tow configuration are up to approximately 15 knots. The Slice is a high-speed variant of the Swath technology with 4 underwater hulls instead of two. Attached to each hull is a strut that extends up to support the main body. Figure 1. shows the profile view of the Slice.[2]

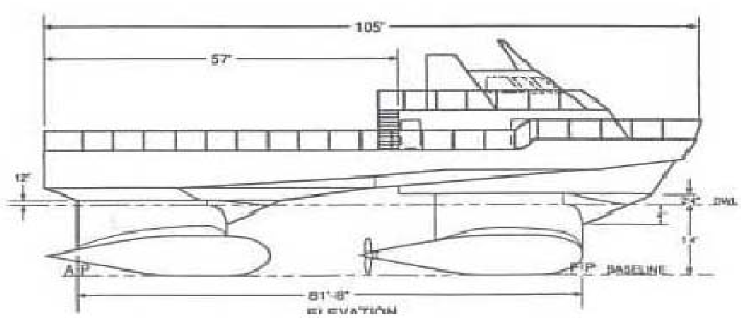

Figure 1. Profile View of SLICE Vessel [From [2]]

The SSP (Semi Submersible Stable Platform) Kaimalino was the world's first high performance open ocean Swath ship. It consists of two parallel torpedo-like hulls. Attached to the hulls are two streamlined struts. The struts extend above the water surface and support the main body. The Kaimalino also has stabilizing fins attached near the aft end of each hull. Figure 2 shows a profile of the SSP Kaimalino. [2]

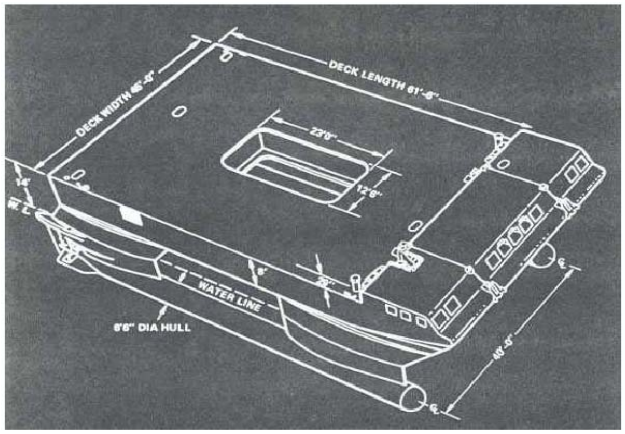

Figure 2. The Profile of the SSP KAIMALINO [From: [2]]

The general approach adopted in this thesis is as follows:

1. First we will use the maneuvering equations of motion in the horizontal plane for both leading and trailing ship.

2. Coupling between the two ships will be provided through the towline. Hydrodynamic coupling arising from radiation and diffraction effects will be neglected. This can be easily incorporated into the analysis once the effect of such hydrodynamic coupling on the hydrodynamic coefficients is established.

3. The system will be linearized in order to study its stability properties. At the end of this part of study identification of regions of stability is expected.

4. A nonlinear analysis will be performed for identifying mechanism of loss of stability. The response of the system will be analyzed with the help of both the linear and nonlinear studies.

The hydrodynamic coefficients of both ships used in this study are provided from reference [2].

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II. PROBLEM FORMULATION AND STABILITY ANALYSIS

A. EQUATIONS OF MOTION

In previous studies, several different values for tension and for length of the connection between the two ships were tried for the stability analysis of the ship towing. [Gokce, March 2002] For every instant, a zero eigenvalue was found. This means the system cannot reach to stable condition. This zero eigenvalue was attributed to the lack of directional stabilizing effects on the leading ship. When the signs of the towline restoring forces and the moments are investigated, it can be seen that towing force has a destabilizing effect on the towing ship. On the other hand it has a stabilizing effect on the towed ship. The ships cannot show directional stability in the horizontal plane, therefore a rudder control law must be applied to the equations of motion.

The mathematical model of the surge, sway and yaw maneuvering equations of motion in dimensionless form is shown below. The subscript 1 refers to towing (leading) ship and the subscript 2 refers to towed (trailing) ship. In these equations u is the surge velocity, v is the sway velocity, R is the resistance of the vessel moving through body of water, and T is the tension in the connection (rope, cable, or some other mechanism) between the two towing ships. The rudder angle is denoted by δ, the heading angle is ψ, and γ is the geometric angle depicted in Figure 3.The equations of motion are:

$$(m_2 - Y_{\dot{v}2})\dot{v}_2 - Y_{\dot{r}2}\dot{r}_2 = -m_2 r_2 + Y_{v2}v_2 + Y_{r2}r_2 - T\sin(\psi_2 + \gamma) \tag{1}$$

$$(I_{Z2} - N_{\dot{r}2})\dot{r}_2 - N_{\dot{v}2}\dot{v}_2 = N_{v2}v_2 + N_{r2}r_2 - Tx_{p2}\sin(\psi_2 + \gamma) \tag{2}$$

$$(m_1 - Y_{\dot{v}1})\dot{v}_1 - Y_{\dot{r}1}r_1 = -m_1 r_1 + Y_{v1}v_1 + Y_{r1}r_1 + T\sin(\psi_1 + \gamma) + Y_\delta \delta \tag{3}$$

$$(I_{Z1} - N_{\dot{r}1})\dot{r}_1 - N_{\dot{v}1}\dot{v}_1 = N_{v1}v_1 + N_{r1}r_1 - Tx_{p1}\sin(\psi_1 + \gamma) + N_\delta \delta \tag{4}$$

where

$$\delta = -k_\psi \psi_1 - k_v v_1 - k_r r_1 - k_y y_1 \tag{5}$$

To study stability of this system, the equations must be linearized. We are dealing with small heading angles, so the small angle assumption can be made.

$$\sin\psi_{1,2} = \psi_{1,2} \tag{6}$$

$$\cos\psi_{1,2} = 1 \tag{7}$$

Another assumption we can make is for the velocities of the ships:

$$u_1 = u_2 = 1 \tag{8}$$

Therefore, the linearized set of equations of motion

$$(m_2 - Y_{\dot{v}2})\dot{v}_2 - Y_{\dot{r}2}\dot{r}_2 = -m_2 r_2 + Y_{v2}v_2 + Y_{r2}r_2 - T(\psi_2 + \gamma) \tag{9}$$

$$(I_{Z2} - N_{\dot{r}2})\dot{r}_2 - N_{\dot{v}2}\dot{v}_2 = N_{v2}v_2 + N_{r2}r_2 - Tx_{p2}(\psi_2 + \gamma) \tag{10}$$

$$(m_1 - Y_{\dot{v}1})\dot{v}_1 - Y_{\dot{r}1}r_1 = -m_1 r_1 + Y_{v1}v_1 + Y_{r1}r_1 + T(\psi_1 + \gamma) + Y_\delta \delta \tag{11}$$

$$(I_{Z1} - N_{\dot{r}1})\dot{r}_1 - N_{\dot{v}1}\dot{v}_1 = N_{v1}v_1 + N_{r1}r_1 - Tx_{p1}(\psi_1 + \gamma) + N_\delta \delta \tag{12}$$

The kinematics and the geometry relations of the system are described by equations (13) and (14).

$$\dot{y}_1 = u_1 \sin\psi_1 + v_1 \cos\psi_1 \tag{13}$$

$$\dot{y}_2 = u_2 \sin\psi_2 + v_2 \cos\psi_2 \tag{14}$$

If we define

$$\bar{y} = y_2 - y_1 \tag{15}$$

then

$$\dot{\bar{y}} = \dot{y}_2 - \dot{y}_1 \tag{16}$$

These geometric relations are explained in Figure 3.

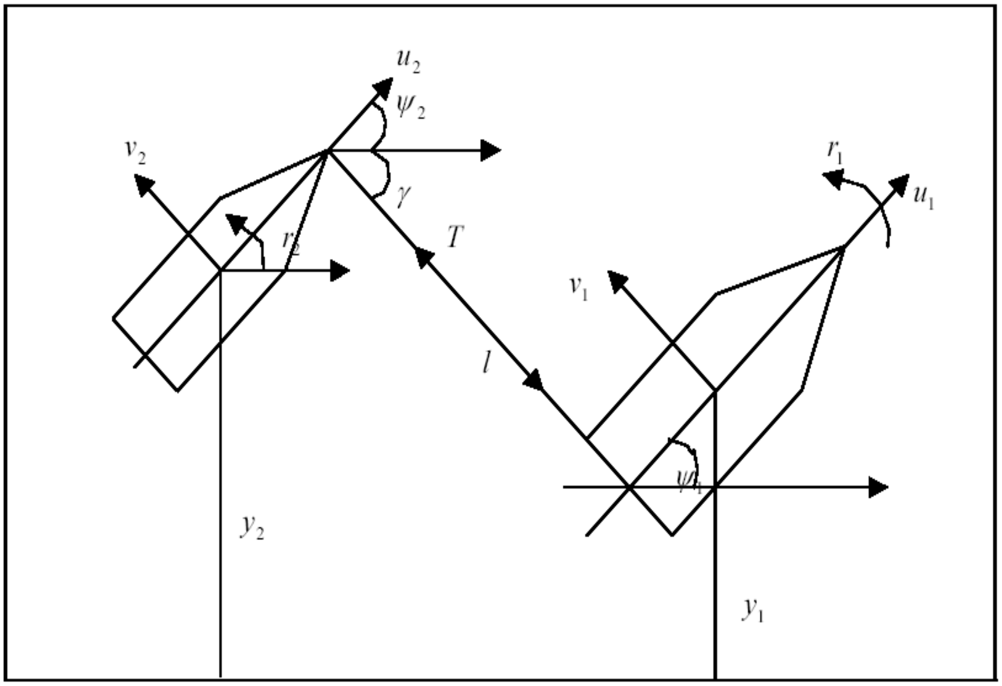

Figure 3. Geometry of Towing Ships

When the equation (13) and equation (14) are substituted into equation (16), we get

$$\dot{\bar{y}} = u_2 \sin\psi_2 + v_2 \cos\psi_2 - u_1 \sin\psi_1 - v_1 \cos\psi_1 \tag{17}$$

From the Figure 3 it can be seen that;

$$\sin\gamma = \frac{y_2 + x_{p2}\sin\psi_2 - (y_1 - x_{p1}\sin\psi_1)}{l} \tag{18}$$

which can be rewritten as

$$\sin\gamma = \frac{\bar{y}}{l} + \frac{1}{l}\left(x_{p2}\sin\psi_2 + x_{p1}\sin\psi 1\right) \tag{19}$$

Using the small angle assumption once more we get

$$\dot{y}_1 = \psi_1 + v_1 \tag{20}$$

$$\dot{y}_2 = \psi_2 + v_2 \tag{21}$$

$$\sin\gamma = \frac{y_2 + x_{p2}\psi_2 - (y_1 - x_{p1}\psi_1)}{l} \tag{22}$$

The next step is to convert these equations into a matrix form.

$$\dot{v}_2 = A_{2vv}v_2 + A_{2vr}r_2 + B_{2v}(\psi_2 + \gamma) \tag{23}$$

$$\dot{r}_2 = A_{2rv}v_2 + A_{2rr}r_2 + B_{2r}(\psi_2 + \gamma) \tag{24}$$

$$\dot{v}_1 = A_{1vv}v_1 + A_{1vr}r_1 + B_{1v}(\psi_1 + \gamma) + C_{1v}\delta \tag{25}$$

$$\dot{r}_1 = A_{1rv}v_1 + A_{1rr}r_1 + B_{1r}(\psi_1 + \gamma) + C_{1r}\delta \tag{26}$$

The coefficients in equations (24), (25), (25) and (26) are found after the following mathematical steps:

$$A_{2vv} = [(I_{z2} - N_{\dot{r}2})Y_{v2} + (N_{v2}Y_{\dot{r}2})] \div [(m_2 - Y_{\dot{v}2})(I_{z2} - N_{\dot{r}2}) - N_{\dot{v}2}Y_{\dot{r}2}] \tag{27}$$

$$A_{2vr} = [(Y_{r2} - m_2)(I_{z2} - N_{\dot{r}2}) + (N_{r2}Y_{\dot{r}2})] \div [(m_2 - Y_{\dot{v}2})(I_{z2} - N_{\dot{r}2}) - N_{\dot{v}2}Y_{\dot{r}2}] \tag{28}$$

$$B_{2v} = [-(I_{z2} - N_{\dot{r}2})T - (Y_{\dot{r}2}Tx_{p2})] \div [(m_2 - Y_{\dot{v}2})(I_{z2} - N_{\dot{r}2}) - N_{\dot{v}2}Y_{\dot{r}2}] \tag{29}$$

$$A_{2rv} = [(Y_{v2}N_{\dot{v}2}) + N_{v2}(m_2 - Y_{\dot{v}2})] \div [(m_2 - Y_{\dot{v}2})(I_{z2} - N_{\dot{r}2}) - N_{\dot{v}2}Y_{\dot{r}2}] \tag{30}$$

$$A_{2rr} = [(Y_{r2} - m_2)N_{\dot{v}2} + N_{r2}(m_2 - Y_{\dot{v}2})] \div [(m_2 - Y_{\dot{v}2})(I_{z2} - N_{\dot{r}2}) - N_{\dot{v}2}Y_{\dot{r}2}] \tag{31}$$

$$B_{2r} = [-TN_{\dot{v}2} - Tx_{p2}(m_2 - Y_{\dot{v}2})] \div [(m_2 - Y_{\dot{v}2})(I_{z2} - N_{\dot{r}2}) - N_{\dot{v}2}Y_{\dot{r}2}] \tag{32}$$

$$A_{1vv} = [(I_{z1} - N_{\dot{r}1})Y_{v1} + (N_{v1}Y_{\dot{r}1})] \div [(m_1 - Y_{\dot{v}1})(I_{z1} - N_{\dot{r}1}) - N_{\dot{v}1}Y_{\dot{r}1}] \tag{33}$$

$$A_{1vr} = [(Y_{r1} - m_1)(I_{z1} - N_{\dot{r}1}) + (N_{r1}Y_{\dot{r}1})] \div [(m_1 - Y_{\dot{v}1})(I_{z1} - N_{\dot{r}1}) - N_{\dot{v}1}Y_{\dot{r}1}] \tag{34}$$

$$B_{1v} = [(I_{z1} - N_{\dot{r}1})T - (Y_{\dot{r}1}Tx_{p2})] \div [(m_1 - Y_{\dot{v}1})(I_{z1} - N_{\dot{r}1}) - N_{\dot{v}1}Y_{\dot{r}1}] \tag{35}$$

$$A_{1rv} = \left[\left(Y_{v1} N_{\dot{v}1} \right) + N_{v1} \left(m_1 - Y_{\dot{v}1} \right) \right] \div \left[\left(m_1 - Y_{\dot{v}1} \right) \left(I_{z1} - N_{\dot{r}1} \right) - N_{\dot{v}1} Y_{\dot{r}1} \right] \tag{36}$$

$$A_{1rr} = \left[\left(Y_{r1} - m_1 \right) N_{\dot{v}1} + N_{r1} \left(m_1 - Y_{\dot{v}1} \right) \right] \div \left[\left(m_1 - Y_{\dot{v}1} \right) \left(I_{z1} - N_{\dot{r}1} \right) - N_{\dot{v}1} Y_{\dot{r}1} \right] \tag{37}$$

$$B_{1r} = \left[TN_{\dot{v}1} + Tx_{p1} \left(m_1 - Y_{\dot{v}1} \right) \right] \div \left[\left(m_1 - Y_{\dot{v}1} \right) \left(I_{z1} - N_{\dot{r}1} \right) - N_{\dot{v}1} Y_{\dot{r}1} \right] \tag{38}$$

$$C_{1v} = \left[\left(I_{z1} - N_{\dot{r}1} \right) Y_{\delta} + N_{\delta} Y_{\dot{r}1} \right] \div \left[\left(m_1 - Y_{\dot{v}1} \right) \left(I_{z1} - N_{\dot{r}1} \right) - N_{\dot{v}1} Y_{\dot{r}1} \right] \tag{39}$$

$$C_{1r} = \left[Y_{\delta} N_{\dot{v}1} + N_{\delta} \left(m_1 - Y_{\dot{v}1} \right) \right] \div \left[\left(m_1 - Y_{\dot{v}1} \right) \left(I_{z1} - N_{\dot{r}1} \right) - N_{\dot{v}1} Y_{\dot{r}1} \right] \tag{40}$$

The hydrodynamic coefficients are taken from reference [2] with the exception of Y_{δ} and N_{δ}. There is no data available for the rudder design of the SLICE, so we assumed a ship turning radius of three ship lengths under fifteen degrees of rudder, and calculated the necessary values of the rudder hydrodynamic coefficients to achieve that turning radius.

The control gains $k_{\psi}, k_v,$ k_r and k_y are calculated by standard pole placement techniques by utilizing the Matlab program shown in the appendices.

The system matrix of 8x8 for the stability analysis is as follows:

$$\begin{bmatrix} \dot{v}_2 \\ \dot{r}_2 \\ \dot{v}_1 \\ \dot{r}_1 \\ \dot{y}_2 \\ \dot{y}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_1 \end{bmatrix} = \left[A \right] \begin{bmatrix} v_2 \\ r_2 \\ v_1 \\ r_1 \\ y_2 \\ y_1 \\ \psi_2 \\ \psi_1 \end{bmatrix} \tag{41}$$

where

$$
[A]=\begin{bmatrix}
A_{2vv} & A_{2vr} & 0 & 0 & \frac{B_{2v}}{l} & -\frac{B_{2v}}{l} & B_{2v}+\frac{B_{2v}x_{p2}}{l} & \frac{B_{2v}x_{p1}}{l} \\
A_{2rv} & A_{2rr} & 0 & 0 & \frac{B_{2r}}{l} & -\frac{B_{2r}}{l} & B_{2r}+\frac{B_{2r}x_{p2}}{l} & \frac{B_{2r}x_{p1}}{l} \\
0 & 0 & A_{1vv}-C_{1v}k_v & A_{1vr}-C_{1v}k_r & \frac{B_{1v}}{l} & -\frac{B_{1v}}{l}-C_{1v}k_y & \frac{B_{1v}x_{p2}}{l} & B_{1v}+\frac{B_{1v}x_{p1}}{l}-C_{1v}k_{\psi} \\
0 & 0 & A_{1rv}-C_{1r}k_v & A_{1rr}-C_{1r}k_r & \frac{B_{1r}}{l} & -\frac{B_{1r}}{l}-C_{1r}k_y & \frac{B_{1r}x_{p2}}{l} & B_{1r}+\frac{B_{1r}x_{p1}}{l}-C_{1r}k_{\psi} \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{42}
$$

Once the matrices are formed, we can predict the stability of the system by an eigenvalue analysis. This analysis is outlined in the next section.

B. DEGREE OF STABILITY

The critical eigenvalue of the system is the one that has largest real part. This largest real part is the degree of the stability of the system. If we have a negative real part it means we have stable system. On the other hand the eigenvalue with a positive real part indicates that the system is unstable.

C. REGIONS OF STABILITY

The results can be summarized in a single graph designating regions of stability and instability. This graph is shown in Figure 4, which was produced by utilizing the Matlab code shown in Appendix A. The graph shows the critical value of the control time constant for stability versus the towline length, and is parameterized by the tension in the towline. Combinations of (T_C,L,T) below the corresponding curve will yield stable response, while combinations that are located above the curve will result in system instability.

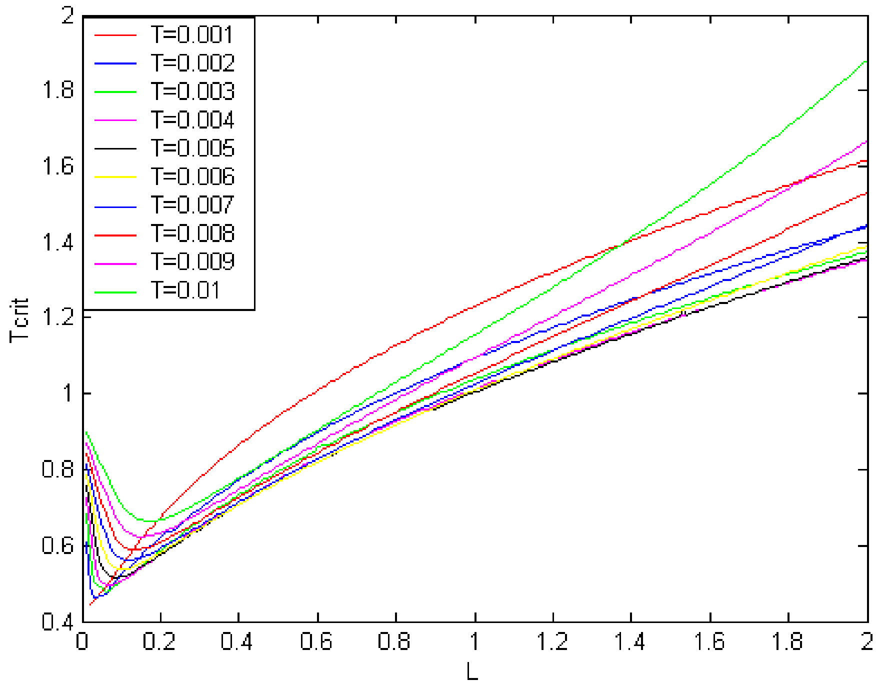

Figure 4. The Stability Region

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III. NONLINEAR STABILITY ANALYSIS

A. HOPF BIFURCATION

In all cases of the loss of stability that we defined in the previous chapter, a pair of complex conjugate eigenvalues has zero real parts. This is called Hopf Bifurcation and is accompanied by periodic solutions (limited cycles). The Hopf Bifurcation occurs when a pair of complex conjugate eigenvalues crosses transversally the imaginary axis. When this happens, the system will deviate from a steady-state solution in an oscillatory manner. This deviation can be either supercritical or subcritical. For the supercritical situation, stable limit cycles form after straight-line stability is lost. Periodic solutions appear after the initial loss of stability and the resulting periodic solutions are stable. For the subcritical cases, the periodic solutions appear before the initial loss of stability and the resulting periodic solutions are unstable. In such a case, the domain of attraction of the equilibrium point becomes progressively smaller as the critical point is reached, and

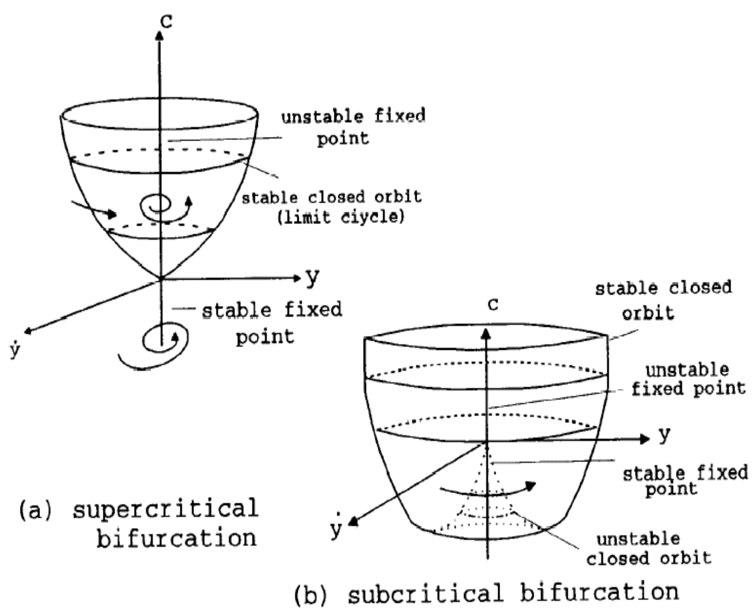

Figure 5. Supercritical and Subcritical Poincare-Andronov-Hopf Bifurcations

in fact it shrinks to zero at the critical point. Therefore, the equilibrium point is unstable under finite disturbances and in practice it may lose its stability even though the linear analysis is predicting it to be stable. Classification of the nature of these bifurcations is the subject of this chapter. In order to evaluate the stability of the limit cycles of Hopf Bifurcations, a nonlinear analysis is needed to be performed.

B. FORMULATION

In the previous chapter, in order to get the degrees of stability and the regions of stability, the equations of motions were linearized. In this way, the linear stability of the ship towing was studied. In this chapter the nonlinear dynamics of the system will be studied.

The nonlinear forms of the equations were described before:

$$(m_2 - Y_{\dot{v}2})\dot{v}_2 - Y_{\dot{r}2}\dot{r}_2 = -m_2 r_2 + Y_{v2}v_2 + Y_{r2}r_2 - T\sin(\psi_2 + \gamma) \tag{1}$$

$$(I_{Z2} - N_{\dot{r}2})\dot{r}_2 - N_{\dot{v}2}\dot{v}_2 = N_{v2}v_2 + N_{r2}r_2 - Tx_{p2}\sin(\psi_2 + \gamma) \tag{2}$$

$$(m_1 - Y_{\dot{v}1})\dot{v}_1 - Y_{\dot{r}1}\dot{r}_1 = -m_1 r_1 + Y_{v1}v_1 + Y_{r1}r_1 + T\sin(\psi_1 + \gamma) + Y_\delta \delta \tag{3}$$

$$(I_{Z1} - N_{\dot{r}1})\dot{r}_1 - N_{\dot{v}1}\dot{v}_1 = N_{v1}v_1 + N_{r1}r_1 - Tx_{p1}\sin(\psi_1 + \gamma) + N_\delta \delta \tag{4}$$

When we write these equations in the matrix form; they are as follows:

$$\dot{v}_2 = A_{2vv}v_2 + A_{2vr}r_2 + B_{2v}\sin\left(\psi_2 + \gamma\right) \tag{43}$$

$$\dot{r}_2 = A_{2rv}v_2 + A_{2rr}r_2 + B_{2r}\sin\left(\psi_2 + \gamma\right) \tag{44}$$

$$\dot{v}_1 = A_{1vv}v_1 + A_{1vr}r_1 + B_{1v}\sin\left(\psi_1 + \gamma\right) + C_{1v}\delta \tag{45}$$

$$\dot{r}_1 = A_{1rv}v_1 + A_{1rr}r_1 + B_{1r}\sin\left(\psi_1 + \gamma\right) + C_{1r}\delta \tag{46}$$

$$\dot{\bar{y}} = u_2\sin\psi_2 + v_2\cos\psi_2 - u_1\sin\psi_1 - v_1\cos\psi_1 \tag{47}$$

The first step for the nonlinear dynamics analysis is to expand the γ term in Taylor series up to the third order. We have:

$$\sin\gamma = \frac{y_2 + x_{p2}\sin\psi_2 - \left(y_1 - x_{p1}\sin\psi_1\right)}{l} \tag{18}$$

by utilizing the $\sin\psi$ taylor expansion:

$$\sin\psi = \psi - \frac{1}{6}\psi^3 \tag{48}$$

$$\gamma = 1/l\left[y_2 + x_{p2}\left(\psi_2 - \frac{1}{6}\psi_2^3\right) - y_1 + x_{p1}\left(\psi_1 - \frac{1}{6}\psi_1^3\right)\right] \tag{49}$$

This alpha value is applied to the equations (43), (44), (45), (46). In order to calculate the nonlinear dynamics coefficient, we have to form a matrix of g values. By these we refer to the nonlinear terms in the equations of motion up to third order. These g (x) values are obtained from the equations (50), (51), (52), (53), (54).

$$\dot{v}_2 = A_{2vv}v_2 + A_{2vr}r_2 + \underbrace{B_{2v}\sin\left(\psi_2 + \gamma\right)}_{g_1(x)} \tag{50}$$

$$\dot{r}_2 = A_{2rv}v_2 + A_{2rr}r_2 + \underbrace{B_{2r}\sin\left(\psi_2 + \gamma\right)}_{g_2(x)} \tag{51}$$

$$\dot{v}_1 = A_{1vv}v_1 + A_{1vr}r_1 + \underbrace{B_{1v}\sin\left(\psi_1 + \gamma\right) + C_{1v}\delta}_{g_3(x)} \tag{52}$$

$$\dot{r}_1 = A_{1rv}v_1 + A_{1rr}r_1 + \underbrace{B_{1r}\sin\left(\psi_1 + \gamma\right) + C_{1r}\delta}_{g_4(x)} \tag{53}$$

$$\dot{y} = \underbrace{\sin\psi_2 + v_2\cos\psi_2 - \sin\psi_1 - v_1\cos\psi_1}_{g_5(x)} \tag{54}$$

With these g (x) values a matrix of 1×7 is formed:

$$g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \\ g_4(x) \\ g_5(x) \\ g_6(x) \\ g_7(x) \end{bmatrix} \tag{55}$$

Next step is to apply the Taylor series expansion for the $\sin(\psi + \gamma)$ part of equations (50), (51), (52), (53) and keep the nonlinear terms up to third order. The terms $g_6(x)$ and $g_7(x)$ are equal to zero. In order to account for rudder angle saturation we substituted the discontinuous saturation function with a smooth function based on the hyperbolic tangent of the rudder angle. Since we are dealing with nonlinear analysis, the nonlinear terms up to third order are kept and the following coefficients are obtained.

$$g_1 = B_{2v}\{\frac{1}{6L^3}x_{p2}^3\psi_2^3 - \frac{1}{2L^2}y_2^2\psi_2 + \frac{1}{2L^3}y_1y_2^2 - \frac{1}{2L^3}y_1^2y_2 + \frac{1}{L}x_{p2}\psi_2 - \frac{1}{2L^2}y_1^2\psi_2$$

$$-\frac{1}{2L^3}x_{p1}^2x_{p2}\psi_1^2\psi_2 - \frac{1}{2L^3}x_{p1}x_{p2}^2\psi_1\psi_2^2 + \frac{1}{12L^3}x_{p1}^2x_{p2}\psi_1^2\psi_2^3 + \frac{1}{L^3}y_1y_2x_{p2}\psi_2 - \frac{1}{6L^3}y_1y_2x_{p2}\psi_2^3$$

$$+\frac{1}{L^3}y_1y_2x_{p1}\psi_1 - \frac{1}{6L^3}y_1y_2x_{p2}\psi_2^3 + \frac{1}{L^3}y_1x_{p1}x_{p2}\psi_1\psi_2 - \frac{1}{6L^3}y_1x_{p1}x_{p2}\psi_1^3\psi_2$$

$$-\frac{1}{6L^3}y_1x_{p1}x_{p2}\psi_1\psi_2^3 + \frac{1}{36L^3}y_1x_{p1}x_{p2}\psi_1^3\psi_2^3 + \frac{1}{6L^3}y_2x_{p1}x_{p2}\psi_1^3\psi_2 + \frac{1}{6L^3}y_2x_{p1}x_{p2}\psi_1\psi_2^3$$

$$-\frac{1}{36L^3}y_2x_{p1}x_{p2}\psi_1^3\psi_2^3 - \frac{1}{L^3}y_2x_{p1}x_{p2}\psi_1\psi_2 - \frac{1}{2L}x_{p1}\psi_1\psi_2^2 + \frac{1}{12L}x_{p1}\psi_1^3\psi_2^2 - \frac{1}{L^2}y_2x_{p2}\psi_2^2$$

$$+\frac{1}{L^2} y_1 x_{p2}\psi_2^2 + \frac{1}{L^2} y_1 y_2\psi_2 - \frac{1}{2L^2} x_{p1}^2\psi_1^2\psi_2 - \frac{1}{2L} y_2\psi_2^2 + \frac{1}{2L} y_1\psi_2^2 - \frac{1}{2L^2} x_{p2}^2\psi_2^3$$

$$+\frac{1}{6L^2} x_{p1}x_{p2}\psi_1^3\psi_2^2 - \frac{1}{L^2} x_{p1}x_{p2}\psi_1\psi_2^2 - \frac{1}{L^2} y_2 x_{p1}x_{p2}\psi_1\psi_2 + \frac{1}{6L^2} y_2 x_{p1}\psi_1^3\psi_2 + \frac{1}{L^2} y_1 x_{p1}\psi_1\psi_2$$

$$-\frac{1}{6L^2} y_1 x_{p1}\psi_1^3\psi_2 - \frac{1}{2L^3} y_2 x_{p2}^2\psi_2^2 + \frac{1}{2L^3} y_1 x_{p1}^2\psi_1^2 + \frac{1}{12L^3} y_2^2 x_{p2}\psi_2^3 - \frac{1}{2L^3} y_2^2 x_{p2}\psi_2$$

$$-\frac{1}{2L^3} y_2^2 x_{p1}\psi_1 + \frac{1}{12L^3} y_2^2 x_{p1}\psi_1^3 + \frac{1}{2L^3} y_1 x_{p2}^2\psi_2^2 - \frac{1}{2L^3} y_1^2 x_{p2}\psi_2 + \frac{1}{12L^3} y_1^2 x_{p2}\psi_2^3$$

$$-\frac{1}{2L^3} y_1^2 x_{p1}\psi_1 + \frac{1}{12L^3} y_1^2 x_{p1}\psi_1^3 - \frac{1}{2L^3} y_2 x_{p1}^2\psi_1^2 - \frac{1}{6}\psi_2^3 - \frac{2}{3L} x_{p2}\psi_2^3 + \frac{1}{L} x_{p1}\psi_1 - \frac{1}{6L} x_{p1}\psi_1^3$$

$$-\frac{1}{6L^3} x_{p1}^3\psi_1^3 + \frac{1}{6L^3} y_1^3 + \frac{1}{12L^3} x_{p1}x_{p2}^2\psi_1^3\psi_2^2 - \frac{1}{6L^3} y_2^3\}$$

$$g_2 = B_{2r}\{\frac{1}{36L^3} y_1 x_{p1}x_{p2}\psi_1^3\psi_2^3 + \frac{1}{12L^3} x_{p1}^2 x_{p2}\psi_1^2\psi_2^3 + \frac{1}{6L^3} y_1^3 - \frac{1}{2L} y_2\psi_2^2 - \frac{1}{2L^2} y_2^2\psi_2$$

$$+\frac{1}{2L} y_1\psi_2^2 - \frac{1}{2L^2} x_{p1}^2\psi_1^2\psi_2 + \frac{1}{6L^2} y_2 x_{p1}\psi_1^3\psi_2 - \frac{1}{L^2} y_2 x_{p1}\psi_1\psi_2 + \frac{1}{L^2} y_1 y_2\psi_2$$

$$+\frac{1}{6L^2} x_{p1}x_{p2}\psi_1^3\psi_2^2 - \frac{1}{L^2} x_{p1}x_{p2}\psi_1\psi_2^2 - \frac{1}{6}\psi_2^3 - \frac{1}{2L^2} y_1^2\psi_2 + \frac{1}{L^3} y_1 x_{p1}x_{p2}\psi_1\psi_2$$

$$-\frac{1}{6L^3} y_1 x_{p1}x_{p2}\psi_1^3\psi_2 - \frac{1}{2L^3} x_{p1}^2 x_{p2}\psi_1^2\psi_2 - \frac{1}{2L^3} x_{p1}x_{p2}^2\psi_1\psi_2^2 + \frac{1}{12L^3} x_{p1}x_{p2}^2\psi_1^3\psi_2^2$$

$$-\frac{1}{6L^3} y_1 x_{p1}x_{p2}\psi_1\psi_2^3 - \frac{1}{2L^3} y_1^2 x_{p2}\psi_2 + \frac{1}{12L^3} y_1^2 x_{p2}\psi_2^3 - \frac{1}{2L^3} y_2 x_{p1}^2\psi_1^2 - \frac{1}{2L^3} y_2^2 x_{p2}\psi_2$$

$$+\frac{1}{2L^3} y_1 x_{p2}^2\psi_2^2 + \frac{1}{L^2} y_1 x_{p2}\psi_2^2 - \frac{1}{2L^3} y_2^2 x_{p1}\psi_1 + \frac{1}{12L^3} y_2^2 x_{p1}\psi_1^3 + \frac{1}{12L^3} y_2^2 x_{p2}\psi_2^3$$

$$-\frac{1}{2L} x_{p1}\psi_1\psi_2^2 - \frac{1}{2L^3} y_2 x_{p2}^2\psi_2^2 - \frac{1}{6L^2} y_1 x_{p1}\psi_1^3\psi_2 + \frac{1}{L^2} y_1 x_{p1}\psi_1\psi_2 - \frac{1}{6L} x_{p1}\psi_1^3$$

$$+\frac{1}{L}x_{p1}\psi_1+\frac{1}{L}x_{p2}\psi_2-\frac{1}{6L^3}y_1y_2x_{p1}\psi_1^3-\frac{1}{L^3}y_2x_{p1}x_{p2}\psi_1\psi_2+\frac{1}{6L^3}y_2x_{p1}x_{p2}\psi_1^3\psi_2$$

$$+\frac{1}{6L^3}y_2x_{p1}x_{p2}\psi_1\psi_2^3-\frac{1}{36L^3}y_2x_{p1}x_{p2}\psi_1^3\psi_2^3-\frac{1}{2L^3}y_1^2y_2+\frac{1}{2L^3}y_1y_2^2-\frac{1}{6L^3}x_{p2}^3\psi_2^3$$

$$-\frac{1}{6L^3}y_2^3-\frac{1}{2L^2}x_{p2}^2\psi_2^3+\frac{1}{L^3}y_1y_2x_{p2}\psi_2-\frac{1}{6L^3}y_1y_2x_{p2}\psi_2^3+\frac{1}{L^3}y_1y_2x_{p1}\psi_1$$

$$-\frac{1}{6L^3}x_{p1}^3\psi_1^3-\frac{1}{2L^3}y_1^2x_{p1}\psi_1+\frac{1}{12L^3}y_1^2x_{p1}\psi_1^3+\frac{1}{2L^3}y_1x_{p1}^2\psi_1^2-\frac{2}{3L}x_{p2}\psi_2^3$$

$$+\frac{1}{12L}x_{p1}\psi_1^3\psi_2^2-\frac{1}{L^2}y_2x_{p2}\psi_2^2$$

$$g_3=B_{1v}\{-\frac{1}{6L^3}y_2^3+\frac{1}{6L^3}y_1^3-\frac{1}{2L^2}y_2^2\psi_1-\frac{1}{2L^2}y_1^2\psi_1-\frac{1}{6L^3}x_{p2}^3\psi_2^3+\frac{1}{2L}y_1\psi_1^2$$

$$+\frac{1}{2L}y_2\psi_1^2+\frac{1}{6L^3}y_2x_{p1}x_{p2}\psi_1\psi_2^3-\frac{1}{L^2}y_2x_{p1}\psi_1^2-\frac{1}{2L}x_{p2}\psi_1^2\psi_2+\frac{1}{12L}x_{p2}\psi_1^2\psi_2^3$$

$$+\frac{1}{L^2}y_1x_{p1}\psi_1^2-\frac{1}{2L^2}x_{p2}^2\psi_1\psi_2^2+\frac{1}{L^2}y_1y_2\psi_1+\frac{1}{12L^3}x_{p1}x_{p2}^2\psi_1^3\psi_2^2-\frac{1}{2L^3}y_2^2x_{p1}\psi_1$$

$$+\frac{1}{12L^3}y_2^2x_{p1}\psi_1^3+\frac{1}{12L^3}y_2^2x_{p2}\psi_2^3-\frac{1}{2L^3}y_2x_{p2}^2\psi_2^2-\frac{1}{2L^3}y_2x_{p1}^2\psi_1^2-\frac{1}{2L^3}y_2^2x_{p2}\psi_2$$

$$+\frac{1}{2L^3}y_1x_{p2}^2\psi_2^2-\frac{1}{2L^3}y-\frac{1}{6L^3}y_1x_{p1}x_{p2}\psi_1\psi_2^3+\frac{1}{12L^3}x_{p1}^2x_{p2}\psi_1^2\psi_2^3$$

$$+\frac{1}{36L^3}y_1x_{p1}x_{p2}\psi_1^3\psi_2^3+\frac{1}{6L^2}x_{p1}x_{p2}\psi_1^2\psi_2^3-\frac{1}{L^2}x_{p1}x_{p2}\psi_1^2\psi_2-\frac{1}{6L^3}y_1y_2x_{p2}\psi_2^3$$

$$+\frac{1}{L^3}y_1y_2x_{p2}\psi_2+\frac{1}{6L^2}y_2x_{p2}\psi_1\psi_2^3+\frac{1}{6L^3}y_2x_{p1}x_{p2}\psi_1^3\psi_{21}^{\;2}x_{p2}\psi_2+\frac{1}{12L^3}y_1^2x_{p2}\psi_2^3$$

$$-\frac{1}{2L^3}y_1^2x_{p1}\psi_1+\frac{1}{12L^3}y_1^2x_{p1}\psi_1^3+\frac{1}{2L^3}y_1x_{p1}^2\psi_1^2$$

$$-\frac{1}{L^2}y_2x_{p2}\psi_1\psi_2+\frac{1}{L^3}y_1y_2x_{p1}\psi_1+\frac{1}{L^2}y_1x_{p2}\psi_1\psi_2-\frac{1}{L^3}y_2x_{p1}x_{p2}\psi_1\psi_2-\frac{1}{6L^3}y_1y_2x_{p1}\psi_1^3$$

$$+\frac{1}{L^3}y_1x_{p1}x_{p2}\psi_1\psi_2-\frac{1}{36L^3}y_2x_{p1}x_{p2}\psi_1^3\psi_2^3-\frac{1}{2L^2}x_{p1}^2\psi_1^3-\frac{2}{3L}x_{p1}\psi_1^3+\frac{1}{2L^3}y_1y_2^2$$

$$-\frac{1}{6L^3}x_{p1}^3\psi_1^3-\frac{1}{6L}x_{p2}\psi_2^3-\frac{1}{2L^3}y_1^2y_2-\frac{1}{6}\psi_1^3-\frac{1}{6L^2}y_1x_{p2}\psi_1\psi_2^3-\frac{1}{2L^3}x_{p1}^2x_{p2}\psi_1^2\psi_2$$

$$-\frac{1}{6L^3}y_1x_{p1}x_{p2}\psi_1^3\psi_2-\frac{1}{2L^3}x_{p1}x_{p2}^2\psi_1\psi_2^2\}-\frac{C_{1v}}{\delta_{sat}^2}\{k_\psi^2k_rr_1\psi_1^2+k_\psi^2k_yy_1\psi_1^2+k_\psi k_v^2v_1^2\psi_1$$

$$+k_\psi k_r^2r_1^2\psi_1+k_\psi k_y^2y_1^2\psi_1+2k_vk_rk_yv_1r_1y_1+k_rk_y^2r_1y_1^2+2k_\psi k_vk_rv_1r_1+2k_\psi k_vk_yv_1y_1\psi_1$$

$$+2k_\psi k_rk_yr_1y_1\psi_1+k_v^2k_rv_1^2r_1+k_v^2k_yv_1^2y_1+k_vk_r^2r_1^2v_1+k_vk_y^2v_1y_1^2+k_r^2k_yr_1^2y_1+\frac{1}{3}k_\psi^3\psi_1^3$$

$$+\frac{1}{3}k_v^3v_1^3+\frac{1}{3}k_r^3r_1^3+\frac{1}{3}k_y^3y_1^3\}$$

$$g_4=B_{1r}\{-\frac{1}{6L^3}y_2^3+\frac{1}{6L^3}y_1^3-\frac{1}{2L^2}y_2^2\psi_1-\frac{1}{2L^2}y_1^2\psi_1-\frac{1}{6L^3}x_{p2}^3\psi_2^3+\frac{1}{2L}y_1\psi_1^2+\frac{1}{2L}y_2\psi_1^2$$

$$+\frac{1}{6L^3}y_2x_{p1}x_{p2}\psi_1\psi_2^3-\frac{1}{L^2}y_2x_{p1}\psi_1^2-\frac{1}{2L}x_{p2}\psi_1^2\psi_2+\frac{1}{12L}x_{p2}\psi_1^2\psi_2^3+\frac{1}{L^2}y_1x_{p1}\psi_1^2$$

$$-\frac{1}{2L^2}x_{p2}^2\psi_1\psi_2^2+\frac{1}{L^2}y_1y_2\psi_1+\frac{1}{12L^3}x_{p1}x_{p2}^2\psi_1^3\psi_2^2-\frac{1}{2L^3}y_2^2x_{p1}\psi_1+\frac{1}{12L^3}y_2^2x_{p1}\psi_1^3$$

$$+\frac{1}{12L^3}y_2^2x_{p2}\psi_2^3-\frac{1}{2L^3}y_2x_{p2}^2\psi_2^2-\frac{1}{2L^3}y_2x_{p1}^2\psi_1^2-\frac{1}{2L^3}y_2^2x_{p2}\psi_2+\frac{1}{2L^3}y_1x_{p2}^2\psi_2^2$$

$$-\frac{1}{2L^3}y_1^2x_{p2}\psi_2+\frac{1}{12L^3}y_1^2x_{p2}\psi_2^3-\frac{1}{2L^3}y_1^2x_{p1}\psi_1+\frac{1}{12L^3}y_1^2x_{p1}\psi_1^3+\frac{1}{2L^3}y_1x_{p1}^2\psi_1^2$$

$$-\frac{1}{6L^3}y_1x_{p1}x_{p2}\psi_1\psi_2^3+\frac{1}{12L^3}x_{p1}^2x_{p2}\psi_1^2\psi_2^3+\frac{1}{36L^3}y_1x_{p1}x_{p2}\psi_1^3\psi_2^3+\frac{1}{6L^2}x_{p1}x_{p2}\psi_1^2\psi_2^3$$

$$-\frac{1}{L^2}x_{p1}x_{p2}\psi_1^2\psi_2-\frac{1}{6L^3}y_1y_2x_{p2}\psi_2^3+\frac{1}{L^3}y_1y_2x_{p2}\psi_2+\frac{1}{6L^2}y_2x_{p2}\psi_1\psi_2^3$$

$$+\frac{1}{6L^3}y_2x_{p1}x_{p2}\psi_1^3\psi_2-\frac{1}{L^2}y_2x_{p2}\psi_1\psi_2+\frac{1}{L^3}y_1y_2x_{p1}\psi_1+\frac{1}{L^2}y_1x_{p2}\psi_1\psi_2-\frac{1}{L^3}y_2x_{p1}x_{p2}\psi_1\psi_2-\frac{1}{6L^3}y_1y_2x_{p1}\psi_1^3$$

$$+\frac{1}{L^3}y_1x_{p1}x_{p2}\psi_1\psi_2-\frac{1}{36L^3}y_2x_{p1}x_{p2}\psi_1^3\psi_2^3-\frac{1}{2L^2}x_{p1}^2\psi_1^3-\frac{2}{3L}x_{p1}\psi_1^3+\frac{1}{2L^3}y_1y_2^2$$

$$-\frac{1}{6L^3}x_{p1}^3\psi_1^3-\frac{1}{6L}x_{p2}\psi_2^3-\frac{1}{2L^3}y_1^2y_2-\frac{1}{6}\psi_1^3-\frac{1}{6L^2}y_1x_{p2}\psi_1\psi_2^3-\frac{1}{2L^3}x_{p1}^2x_{p2}\psi_1^2\psi_2$$

$$-\frac{1}{6L^3}y_1x_{p1}x_{p2}\psi_1^3\psi_2-\frac{1}{2L^3}x_{p1}x_{p2}^2\psi_1\psi_2^2\}-\frac{C_{1r}}{\delta_{sat}^2}\{k_\psi^2k_rr_1\psi_1^2+k_\psi^2k_yy_1\psi_1^2+k_\psi k_v^2v_1^2\psi_1$$

$$+k_\psi k_r^2r_1^2\psi_1+k_\psi k_y^2y_1^2\psi_1+2k_vk_rk_yv_1r_1y_1+k_rk_y^2r_1y_1^2+2k_\psi k_vk_rv_1r_1+2k_\psi k_vk_yv_1y_1\psi_1$$

$$+2k_\psi k_rk_yr_1y_1\psi_1+k_v^2k_rv_1^2r_1+k_v^2k_yv_1^2y_1+k_vk_r^2r_1^2v_1+k_vk_y^2v_1y_1^2+k_r^2k_yr_1^2y_1+\frac{1}{3}k_\psi^3\psi_1^3$$

$$+\frac{1}{3}k_v^3v_1^3+\frac{1}{3}k_r^3r_1^3+\frac{1}{3}k_y^3y_1^3\}$$

In order to get g_5, a Taylor series expansion of sine and cosine is used.

$$\sin\psi=\psi-\frac{1}{6}\psi^3$$

$$\cos\psi=1-\frac{1}{2}\psi^2$$

$$g_5=\psi_2-\psi_1-\frac{1}{6}\left(\psi_2^3-\psi_1^3\right)+v_2-v_1-\frac{1}{2}\left(v_2\psi_2^2-v_1\psi_1^2\right)$$

The next step is to transform coordinates to normal form. This is obtained by the using matrix of eigenvectors [m] at the critical point as the coordinate transformation matrix. The new set of coordinates is called [z], so in terms of the original set we have [x]=[m][z]. The only variables of [z] that need to be retained in the analysis of this problem are first two, the remaining variables are expressed in terms of first two

variables through higher order relationships and do not affect the third order expansions that we utilized.

$$\psi_1 = m_{11}z_1 + m_{12}z_2$$

$$v_1 = m_{21}z_1 + m_{22}z_2$$

$$r_1 = m_{31}z_1 + m_{32}z_2$$

$$y_1 = m_{41}z_1 + m_{42}z_2$$

$$\psi_2 = m_{51}z_1 + m_{52}z_2$$

$$v_2 = m_{61}z_1 + m_{62}z_2$$

$$r_2 = m_{71}z_1 + m_{72}z_2$$

$$y_2 = m_{81}z_1 + m_{82}z_2$$

When the above terms are substituted into the equation of motion, we get:

$$L_{11} = B_{2v}\{\frac{1}{6L^3}(m_{41}^3 - m_{81}^3 - m_{51}^3x_{p2}^3 - m_{11}^3x_{p1}^3) - \frac{1}{2L^2}(m_{51}^3x_{p2}^2 + m_{41}^2m_{51} + m_{11}^2m_{51}x_{p1}^2)$$

$$-\frac{1}{6}m_{51}^3 - \frac{2}{3L}m_{51}^3x_{p2} - \frac{1}{6L}m_{11}^3x_{p1} + \frac{1}{2L^3}(m_{41}m_{81}^2 - m_{41}^2m_{81} - m_{51}^2m_{81}x_{p2} + m_{41}m_{51}^2x_{p2}^2$$

$$-m_{11}^2m_{81}x_{p1}^2 - m_{11}m_{81}^2x_{p1} - m_{41}^2m_{51}x_{p2} - m_{11}m_{41}^2x_{p1} + m_{11}^2m_{41}x_{p1}^2 - m_{11}m_{51}^2x_{p2}^2x_{p1}$$

$$-m_{11}^2m_{51}x_{p1}^2x_{p2}) - \frac{1}{L^2}(m_{51}^2m_{81}x_{p2} - m_{41}m_{51}^2x_{p2} - m_{41}m_{51}m_{81} + m_{11}m_{51}m_{81}x_{p1}$$

$$-m_{11}m_{41}m_{51}x_{p1} + m_{11}m_{51}^2x_{p1}x_{p2}) - \frac{1}{2L}(m_{51}^2m_{81} - m_{41}m_{51}^2 + m_{11}m_{51}^2x_{p1}) + \frac{1}{L^3}(m_{41}m_{51}m_{81}x_{p2}$$

$$+m_{11}m_{41}m_{81}x_{p1} - m_{11}m_{51}m_{81}x_{p1}x_{p2} + m_{11}m_{41}m_{51}x_{p1}x_{p2})$$

$$L_{12}=B_{2v}\{\frac{1}{6L^3}(m_{42}^3-m_{82}^3-m_{52}^3x_{p2}^3-m_{12}^3x_{p1}^3)-\frac{1}{2L^2}(m_{52}^3x_{p2}^2-m_{12}^2m_{52}x_{p1}^2)\frac{1}{6}m_{52}^3-\frac{2}{3L}m_{52}^3x_{p2}$$

$$-\frac{1}{6L}m_{12}^3x_{p1}+\frac{1}{2L^3}(m_{42}m_{82}^2-m_{42}^2m_{82}-m_{52}^2m_{82}x_{p2}^2+m_{42}m_{52}^2x_{p2}^2-2m_{12}^2m_{82}x_{p1}^2-m_{12}m_{82}^2x_{p1}$$

$$-m_{42}^2m_{52}x_{p2}-m_{12}m_{42}^2x_{p1}+m_{12}^2m_{42}x_{p1}^2-m_{12}m_{52}^2x_{p1}x_{p2}^2-m_{12}^2m_{52}x_{p1}^2x_{p2})-\frac{1}{L^2}(m_{82}m_{52}^2x_{p2}$$

$$-m_{42}m_{52}^2x_{p2}+2m_{42}^2m_{52}-m_{42}m_{52}m_{82}+m_{12}m_{52}m_{82}x_{p1}-m_{12}m_{52}m_{42}x_{p1}+m_{12}m_{52}^2x_{p1}x_{p2})$$

$$-\frac{1}{2L}(m_{52}^2m_{82}-m_{42}m_{52}^2+m_{12}m_{52}^2x_{p1})+\frac{1}{L^3}(m_{42}m_{52}m_{82}x_{p2}+m_{12}m_{42}m_{82}x_{p1}-m_{12}m_{52}m_{82}x_{p1}x_{p2}$$

$$+m_{12}m_{52}m_{42}x_{p1}x_{p2})$$

$$L_{13}=B_{2v}\{\frac{1}{2L^3}(m_{41}^2m_{42}-m_{81}^2m_{82}-m_{51}^2m_{52}x_{p2}^3-m_{11}^2m_{12}x_{p1}^3+m_{42}m_{81}^2-m_{41}^2m_{82}-m_{51}^2m_{82}x_{p2}^2$$

$$+m_{42}m_{51}^2x_{p2}^2-2m_{11}^2m_{82}x_{p1}^2-m_{12}m_{81}^2x_{p1}-m_{12}m_{41}^2x_{p1}+m_{11}^2m_{42}x_{p1}^2-m_{12}m_{51}^2x_{p1}x_{p2}^2$$

$$-m_{11}^2m_{52}x_{p1}^2x_{p2})-\frac{1}{L^3}(\frac{3}{2}m_{51}^2m_{52}x_{p2}^2-m_{41}m_{81}m_{82}+m_{41}m_{42}m_{81}+m_{51}m_{52}m_{81}x_{p2}^2$$

$$-m_{41}m_{51}m_{52}x_{p2}^2+2m_{11}m_{12}m_{81}x_{p1}^2+m_{11}m_{81}m_{82}x_{p1}+m_{41}m_{42}m_{51}x_{p2}+m_{11}m_{41}m_{42}x_{p1}-m_{11}m_{12}m_{41}x_{p1}^2$$

$$-m_{41}m_{51}m_{82}x_{p2}-m_{42}m_{51}m_{81}x_{p2}-m_{12}m_{41}m_{81}x_{p1}-m_{11}m_{41}m_{82}x_{p1}-m_{11}m_{42}m_{81}+m_{11}m_{52}m_{81}x_{p1}x_{p2}$$

$$+m_{12}m_{51}m_{81}x_{p1}x_{p2}+m_{11}m_{51}m_{82}x_{p1}x_{p2}-m_{11}m_{41}m_{52}x_{p1}x_{p2}-m_{12}m_{41}m_{51}x_{p1}x_{p2}$$

$$-m_{11}m_{42}m_{51}x_{p1}x_{p2}+m_{11}m_{51}m_{52}x_{p1}x_{p2}^2+m_{11}m_{12}m_{51}x_{p1}^2x_{p2})-\frac{1}{2}m_{51}^2m_{52}-\frac{1}{2L}(m_{11}^2m_{12}x_{p1}$$

$$+m_{51}^2m_{82}-m_{42}m_{51}^2+m_{12}m_{51}^2x_{p1})-\frac{1}{L^2}(2m_{51}m_{52}m_{81}x_{p2}+m_{51}^2m_{82}x_{p2}-2m_{41}m_{51}m_{52}x_{p2}$$

$$-m_{42}m_{51}^2x_{p2}+m_{41}m_{42}m_{51}-m_{41}m_{52}m_{81}x_{p2}-m_{41}m_{52}m_{81}-m_{41}m_{51}m_{82}-m_{42}m_{51}m_{81}$$

$$+m_{11}m_{52}m_{81}x_{p1}+m_{12}m_{51}m_{81}x_{p1}+m_{11}m_{51}m_{82}x_{p1}-m_{11}m_{41}m_{52}x_{p1}-m_{12}m_{41}m_{51}x_{p1}$$

$$-m_{11}m_{42}m_{51}x_{p1}+2m_{11}m_{51}m_{52}x_{p1}x_{p2}+m_{12}m_{51}^2x_{p1}x_{p2}+m_{11}m_{12}m_{51}x_{p1}^2)-\frac{1}{L}(m_{51}m_{52}m_{81}$$

$$-m_{41}m_{51}m_{52}+m_{11}m_{51}m_{52}x_{p1})-\frac{1}{2L^2}(m_{41}^2m_{52}+m_{41}^2m_{52}x_{p2}+m_{11}^2m_{52}x_{p1}^2)$$

$$L_{14}=B_{2v}\{\frac{1}{2L^3}(m_{41}m_{42}^2-m_{81}m_{82}^2-m_{51}m_{52}^2x_{p2}^3-m_{11}m_{12}^2x_{p1}^3+m_{41}m_{82}^2-m_{42}^2m_{81}$$

$$-m_{52}^2m_{81}x_{p2}^2+m_{41}m_{52}^2x_{p2}^2-m_{42}^2m_{51}x_{p2}-m_{11}m_{42}^2x_{p1}+m_{12}^2m_{41}x_{p1}^2-m_{11}m_{52}^2x_{p1}x_{p2}^2$$

$$-m_{12}^2m_{51}x_{p1}^2x_{p2})-\frac{1}{L^2}(\frac{3}{2}m_{51}m_{52}^2x_{p2}^2+m_{52}^2m_{81}x_{p2}+2m_{51}m_{52}m_{82}x_{p2}-m_{41}m_{52}^2x_{p2}$$

$$-2m_{42}m_{51}m_{52}x_{p2}-m_{41}m_{52}m_{82}-m_{42}m_{52}m_{81}-m_{42}m_{51}m_{82}+m_{12}m_{52}m_{81}x_{p1}$$

$$+m_{11}m_{52}m_{82}x_{p1}+m_{12}m_{51}m_{82}x_{p1}-m_{12}m_{41}m_{52}x_{p1}-m_{11}m_{42}m_{52}x_{p1}-m_{12}m_{42}m_{51}x_{p1}$$

$$+m_{11}m_{52}^2x_{p1}x_{p2}-2m_{12}m_{51}m_{52}x_{p1}x_{p2}+m_{11}m_{12}m_{52}x_{p1}^2+m_{12}^2m_{51}x_{p1}^2)-\frac{1}{2}m_{51}m_{52}^2$$

$$-\frac{1}{L}(2m_{51}m_{52}^2x_{p2}+m_{51}m_{52}m_{82}-m_{42}m_{51}m_{52}+m_{12}m_{51}m_{52}x_{p1})-\frac{1}{2L}(m_{11}m_{12}^2x_{p1}$$

$$+m_{52}^2m_{81}-m_{41}m_{52}^2+m_{11}m_{52}^2x_{p1})+\frac{1}{L^3}(m_{42}m_{81}m_{82}-m_{41}m_{42}m_{82}-m_{51}m_{52}m_{82}x_{p2}^2$$

$$+m_{42}m_{51}m_{52}x_{p2}^2-m_{41}m_{42}m_{52}x_{p2}-m_{12}m_{41}m_{42}x_{p1}+m_{11}m_{12}m_{42}x_{p1}^2+m_{41}m_{52}m_{82}x_{p2}$$

$$+m_{42}m_{52}m_{81}x_{p2}+m_{42}m_{51}m_{82}x_{p2}+m_{12}m_{41}m_{82}x_{p1}+m_{12}m_{42}m_{81}x_{p1}+m_{11}m_{42}m_{82}x_{p1}$$

$$-m_{12}m_{52}m_{82}x_{p1}x_{p2}-m_{11}m_{52}m_{82}x_{p1}x_{p2}-m_{12}m_{51}m_{82}x_{p1}x_{p2}+m_{12}m_{41}m_{52}x_{p1}x_{p2}$$

$$+m_{11}m_{52}m_{42}x_{p1}x_{p2}+m_{12}m_{42}m_{51}x_{p1}x_{p2}-m_{12}m_{51}m_{52}x_{p1}x_{p2}^2-m_{11}m_{12}m_{52}x_{p1}^2x_{p2})$$

$$-\frac{1}{2L^2}(m_{42}^2 m_{51})\}$$

Following the above background calculations we are now ready to summarize the nonlinear analysis for our system.

C. THE CONTROL LAW

A typical full state feedback control law will be utilized for the nonlinear analysis again. In the below equation δ_0 is the feedback rudder angle for small deviations.

$$\delta_0 = k_\psi \psi_1 + k_v v_1 + k_r r_1 + k_y y_1 \tag{56}$$

In order to capture the effect of rudder saturation, the control law is modeled as:

$$\delta = \delta_{sat} \tanh\left(\frac{\delta_0}{\delta_{sat}}\right) \tag{57}$$

The hyperbolic tangent function is used instead of hard saturation function because of its analyticity properties.

Utilizing the taylor expansion again;

$$\sin\psi = \psi - \frac{1}{6}\psi^3$$

$$\delta = \delta_0 - \frac{1}{3\delta_{sat}^2}\delta_0^3 \tag{58}$$

In the Matlab (Appendix A and B) code for the nonlinear analysis saturation limit on the δ is set to 0.4.

D. INTEGRAL AVERAGING

When the equations of motion are written in the compact form,

$$\dot{X} = f(x), \qquad \dot{X} = [v_2, r_2, v_1, r_1, y_2, y_1, \psi_2, \psi_1]^T \tag{59}$$

The system can be written in the form

$$\dot{X} = AX + g(x) \tag{60}$$

A is the Jacobian matrix of f (x), and g (x) contains all nonlinear term of equations (50), (51), (52), (53), (54).

If we introduce the transformation matrix (T) of eigenvectors of A

$$T^{-1} = [m_{ij}] \qquad i, j = 1,2,3,4,5,6,7,8 \tag{61}$$

$$T^{-1} = [n_{ij}] \qquad i, j = 1,2,3,4,5,6,7,8 \tag{62}$$

linear change of coordinates

$$x = Tz \qquad z = T^{-1}x \tag{63}$$

will transform system (X) into its normal form

$$\dot{z} = T^{-1}ATz + T^{-1}g(Tz) \tag{64}$$

with this transformation we get

$$T^{-1}AT = \begin{bmatrix} 0 & -w & 0 & 0 & 0 & 0 & 0 & 0 \\ -w & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_6 \end{bmatrix} \tag{65}$$

The expressions for g are given by:

$$g = \begin{bmatrix} l_{11}z_1^3 + l_{12}z_2^3 + l_{13}z_1^2 z_2 + l_{14}z_1 z_2^3 \\ l_{21}z_1^3 + l_{22}z_2^3 + l_{23}z_1^2 z_2 + l_{24}z_1 z_2^3 \\ l_{31}z_1^3 + l_{32}z_2^3 + l_{33}z_1^2 z_2 + l_{34}z_1 z_2^3 \\ l_{41}z_1^3 + l_{42}z_2^3 + l_{43}z_1^2 z_2 + l_{44}z_1 z_2^3 \\ l_{51}z_1^3 + l_{52}z_2^3 + l_{53}z_1^2 z_2 + l_{54}z_1 z_2^3 \end{bmatrix} \tag{66}$$

If we substitute equation (63) into (60), we get

$$\dot{z}_1 = -wz_2 + r_{11}z_1^3 + r_{12}z_1^2 z_2 + r_{13}z_1 z_2^2 + r_{14}z_2^3 \tag{67}$$

$$\dot{z}_2 = -wz_1 + r_{21}z_1^3 + r_{22}z_1^2 z_2 + r_{23}z_1 z_2^2 + r_{24}z_2^3 \tag{68}$$

These equations can be written as,

$$\dot{z}_1 = -wz_2 + F_1(z_1, z_2), \tag{69}$$

$$\dot{z}_2 = -wz_1 + F_2(z_1, z_2), \tag{70}$$

Introducing the polar coordinates for the equations (69) and (70),

$$z_1 = R\cos\theta, \qquad z_2 = R\sin\theta \tag{71}$$

we get

$$\dot{R} = F_1(R,\theta)\cos\theta + F_2(R,\theta)\sin\theta \tag{72}$$

$$R\dot{\theta} = wR + F_2(R,\theta)\cos\theta - F_1(R,\theta)\sin\theta \tag{73}$$

Equation (73) gives

$$\dot{R} = P(\theta)R^3 \tag{74}$$

The function $P(\theta)$ is a 2π periodic function in the angular coordinate θ. In order to get the equation with constant coefficients, equation (74) is averaged over one cycle in θ. This way we get

$$\dot{R} = KR^3 \tag{75}$$

and

$$K = \frac{1}{2\pi}\int_0^{\pi} P(\theta).d\theta \tag{76}$$

After making the integral calculations we get the simplified form of the equation (76)

$$K = \frac{1}{8}[3r_{11} + r_{13} + r_{22} + 3r_{24}] \tag{77}$$

The indicated coefficient are given by,

$$\begin{aligned}
\mathrm{r}_{11} &= n_{11}l_{11} + n_{12}l_{21} + n_{13}l_{31} + n_{14}l_{41} + n_{15}l_{51} \\
r_{13} &= n_{11}l_{14} + n_{12}l_{24} + n_{13}l_{34} + n_{14}l_{44} + n_{15}l_{54} \\
r_{22} &= n_{21}l_{13} + n_{22}l_{23} + n_{23}l_{33} + n_{24}l_{43} + n_{25}l_{53} \\
r_{24} &= n_{21}l_{12} + n_{22}l_{22} + n_{23}l_{32} + n_{24}l_{42} + n_{25}l_{52}
\end{aligned} \tag{78}$$

The coefficients n_{ij} are the elements of the inverse of transformation matrix T, which was defined as the matrix of eigenvectors of the linearized matrix A.

Results in terms of the nonlinear stability coefficient K are shown in Figure 6. It can be seen that the short towline lengths result in supercritical bifurcations, while larger lengths result in subcritical bifurcations. The implications of this on system stability under finite disturbances were discussed previously. It should be pointed out that the transition from supercritical to subcritical bifurcations as towline length is increased is dependent on the hydrodynamic characteristics of the two ships and is not necessarily true for all ships. The procedure that we have established in this work should be followed in order to assess the type of corresponding bifurcation.

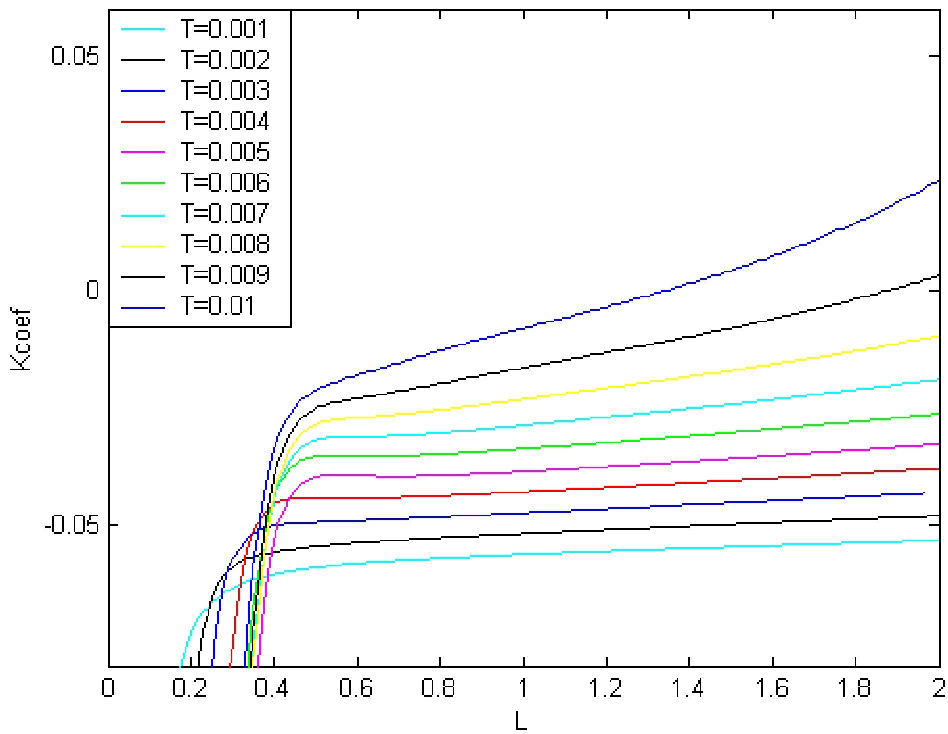

Figure 6. Nonlinear Dynamics Coefficient vs. Towline Length

IV. CONCLUSIONS AND RECOMMENDATIONS

A. CONCLUSIONS

This thesis presented a comprehensive study of the stability region properties and the nonlinear dynamics response of the two-body ship-towing problem. The results of this study can help in establishing rational guidelines, which should be followed in order to ensure stability and safety of close proximity ship towing operations. The main results from this study can be summarized as follows:

1. The traditional towing stability model cannot predict the dynamics of towed ships in close proximity. One of the reasons of this result is excluding the dynamics of the leading ship in the overall analysis.

2. Path stability depends on classical parameters such as tension and towline length, but also on additional terms such as the leading ship's control time constant.

3. For very large values of the towline length, the effect of the leading ship control becomes less pronounced. Therefore, previously reported results in the literature can be considered as a large towline length of the two-body model developed in this work.

4. The primary mechanism of loss of stability is in terms of Hopf Bifurcations to periodic solutions. The bifurcation can be classified as either supercritical or subcritical in terms of a compatible nonlinear stability coefficient.

5. Hopf Bifurcations tend to be supercritical for small values of towline length, while they tend to turn into subcritical as the towline length is increased.

B. RECOMMENDATIONS

Recommendations for continuing studies are the following:

1. Analyze the properties of stability and bifurcations including the towline elasticity into the model.

2. Analyze the stability properties of different vessels and investigate the occurrence of static bifurcations.

APPENDIX A. MATLAB CODE TO FIND THE REGION OF STABILITY

```
% Stability graph - TC vs. L - constant T
%
% Parameters:
%   T       = Nondimensional tension
%   L       = Towline length
%   TC      = Control law time constant
%    xp1      = Attachment point (leading ship - positive forward
of amidships)
%    xp2      = Attachment point (trailing ship - positive aft of
amidships)
%
% Constants
%
u1     =  1;
u2     =  1;
m1     =  0.018078;
m2     =  0.018;
Iz1    =  0.0007;
Iz2    =  0.00069412;
Yv1    = -0.07893;
Yv2    = -0.1183;
Yr1    = -0.004044;
Yr2    = -0.0042;
Nv1    = -0.016428;
Nv2    = -0.0187;
Nr1    = -0.010332;
Nr2    = -0.0176;
Yvdot1 = -0.051328;
Yvdot2 = -0.0184;
Yrdot1 =  0.005617;
Yrdot2 = -0.0011;
Nvdot1 = -0.001945;
Nvdot2 = -0.0008489;
Nrdot1 = -0.00564;
Nrdot2 = -0.0090;
Ydel   =  0.0103;
Ndel   = -0.0051;
indexL = 0;
index  = 0;
xp1    = 0.5;
xp2    = 0.5;
%
% Enter constant T
%
T      = input('T  = ');
%
% Start loop on length
%
for iL =0.1:0.01:2.0;
    indexL     =  indexL+1;
    indexTC    =  0;
    L          =  iL
    L_v(indexL)=  L;
    A3         =  1/L;
```

```
          B3         =  xp1/L;
          C3         =  xp2/L;
          D3         = -1/L;
          %
          % Loop on TC
          %
          for iTC    =0.1:0.01:2.0;
              indexTC=indexTC+1;
              TC     =iTC;
              bpole  = -1/TC;
              TC_v(indexTC)=TC;
              %
              % Setup the matrix coefficients
              %
              A2vv      =      [((Iz2-Nrdot2)*Yv2)+(Nv2*Yrdot2)]/[((m2-
Yvdot2)*(Iz2-Nrdot2))-(Nvdot2*Yrdot2)];
              A2vr  =  [((Yr2-(m2*u2))*(Iz2-Nrdot2))+(Nr2*Yrdot2)]/[((m2-
Yvdot2)*(Iz2-Nrdot2))-(Nvdot2*Yrdot2)];
              B2v        =    [-((Iz2-Nrdot2)*T)-(Yrdot2*T*xp2)]/[((m2-
Yvdot2)*(Iz2-Nrdot2))-(Nvdot2*Yrdot2)] ;
              A2rv      =      [(Yv2*Nvdot2)+(Nv2*(m2-Yvdot2))]/[((m2-
Yvdot2)*(Iz2-Nrdot2))-(Nvdot2*Yrdot2)];
              A2rr  =  [((Yr2-(m2*u2))*Nvdot2)+(Nr2*(m2-Yvdot2))]/[((m2-
Yvdot2)*(Iz2-Nrdot2))-(Nvdot2*Yrdot2)];
              B2r        =    [-(T*Nvdot2)-(T*xp2*(m2-Yvdot2))]/[((m2-
Yvdot2)*(Iz2-Nrdot2))-(Nvdot2*Yrdot2)];
              A1vv      =      [((Iz1-Nrdot1)*Yv1)+(Nv1*Yrdot1)]/[((m1-
Yvdot1)*(Iz1-Nrdot1))-(Nvdot1*Yrdot1)];
              A1vr   =   [(Yr1-m1*u1)*(Iz1-Nrdot1)+(Nr1*Yrdot1)]/[((m1-
Yvdot1)*(Iz1-Nrdot1))-(Nvdot1*Yrdot1)];
              B1v        =    [((Iz1-Nrdot1)*T)-(Yrdot1*T*xp1)]/[((m1-
Yvdot1)*(Iz1-Nrdot1))-(Nvdot1*Yrdot1)];
              C1v       =    [((Iz1-Nrdot1)*Ydel)+(Ndel*Yrdot1)]/[((m1-
Yvdot1)*(Iz1-Nrdot1))-(Nvdot1*Yrdot1)];
              A1rv      =      [(Yv1*Nvdot1)+(Nv1*(m1-Yvdot1))]/[((m1-
Yvdot1)*(Iz1-Nrdot1))-(Nvdot1*Yrdot1)];
              A1rr  =  [((Yr1-(m1*u1))*Nvdot1)+(Nr1*(m1-Yvdot1))]/[((m1-
Yvdot1)*(Iz1-Nrdot1))-(Nvdot1*Yrdot1)];
              B1r         =    [(T*Nvdot1)-(T*xp1*(m1-Yvdot1))]/[((m1-
Yvdot1)*(Iz1-Nrdot1))-(Nvdot1*Yrdot1)];
              C1r        =    [(Ydel*Nvdot1)+(Ndel*(m1-Yvdot1))]/[((m1-
Yvdot1)*(Iz1-Nrdot1))-(Nvdot1*Yrdot1)];
              %
              % Find control gains
              %
              C      = zeros(4);
              C(1,3) = 1;
              C(2,2) = A1vv;
              C(2,3) = A1vr;
              C(3,2) = A1rv;
              C(3,3) = A1rr;
              C(4,1) = 1;
              C(4,2) = 1;
              D      = zeros(4,1);
              D(2,1) = C1v;
              D(3,1) = C1r;
```

```
poles  = [bpole bpole-0.05 bpole-0.10 bpole-0.15];
k      = place(C,D,poles);
Kpsi   = k(1,1);
Kv     = k(1,2);
Kr     = k(1,3);
Ky     = k(1,4);
%
% A matrix
%
A      = zeros(8);
A(1,1) = A2vv;
A(1,2) = A2vr;
A(1,5) = B2v/L;
A(1,6) = -B2v/L;
A(1,7) = B2v+((B2v*xp2)/L);
A(1,8) = (B2v*xp1)/L;
A(2,1) = A2rv;
A(2,2) = A2rr;
A(2,5) = B2r/L;
A(2,6) = -B2r/L;
A(2,7) = B2r+((B2r*xp2)/L);
A(2,8) = (B2r*xp1)/L;
A(3,3) = A1vv-(C1v*Kv);
A(3,4) = A1vr-(C1v*Kr);
A(3,5) = B1v/L;
A(3,6) = -B1v/L-(C1v*Ky);
A(3,7) = (B1v*xp2)/L;
A(3,8) = B1v+((B1v*xp1)/L)-(C1v*Kpsi);
A(4,3) = A1rv-(C1r*Kv);
A(4,4) = A1rr-(C1r*Kr);
A(4,5) = B1r/L;
A(4,6) = -B1r/L-(C1r*Ky);
A(4,7) = (B1r*xp2)/L;
A(4,8) = B1r+((B1r*xp1)/L)-(C1r*Kpsi);
A(5,1) = 1;
A(5,7) = 1;
A(6,3) = 1;
A(6,8) = 1;
A(7,2) = 1;
A(8,4) = 1;
%
% Find eigenvalues
%
B=eig(A);
F=real(B);
H_new=max(F);
%
% Detect if H changes its sign
%
if indexTC==1
    H_old=H_new;
else
    PROD=H_new*H_old;
    if PROD > 0
        %
        % H does not change sign - keep going
```

```
                      %
                      H_old=H_new;
                  else
                      %
                      % H changed its sign - find critical TC by linear
interpolation
                      %
                      index=index+1;
                      TC_crit(index)=-(H_old*TC_v(indexTC)-
H_new*TC_v(indexTC-1))/(H_new-H_old);
                      H_old=H_new;
                  end
              end
          end
      end
      %
      % Plot (L, crit TC) curve
      %
      %plot(L_v,TC_crit),xlabel('L'),ylabel('{T_C}_{\rm
crit}'),title('Constant T'),grid
```

APPENDIX B. MATLAB CODE TO FIND THE NONLINEAR DYNAMICS COEFFICIENT

```
% Nonlinear Dynamics coefficient graph - K vs. L - constant T
%
% Parameters:
%   T       = Nondimensional tension
%   L       = Towline length
%   TC      = Control law time constant
%   xp1     = Attachment point (leading ship - positive forward of
amidships)
%   xp2     = Attachment point (trailing ship - positive aft of
amidships)
%
% Constants
%
u1     =  1;
u2     =  1;
m1     =  0.018078;
m2     =  0.018;
Iz1    =  0.0007;
Iz2    =  0.00069412;
Yv1    = -0.07893;
Yv2    = -0.1183;
Yr1    = -0.004044;
Yr2    = -0.0042;
Nv1    = -0.016428;
Nv2    = -0.0187;
Nr1    = -0.010332;
Nr2    = -0.0176;
Yvdot1 = -0.051328;
Yvdot2 = -0.0184;
Yrdot1 =  0.005617;
Yrdot2 = -0.0011;
Nvdot1 = -0.001945;
Nvdot2 = -0.0008489;
Nrdot1 = -0.00564;
Nrdot2 = -0.0090;
Ydel   =  0.0103;
Ndel   = -0.0051;
indexL = 0;
index  = 0;
xp1    = 0.5;
xp2    = 0.5;
deltasat=0.4;
%
% Enter constant T
%
T      = input('T  = ');
% Start loop on length
%
for iL =1.53:0.01:2.0;
indexL     =  indexL+1;
```

```
indexTC    =  0;
L          =  iL
L_v(indexL)=  L;
A3         =  1/L;
B3         =  xp1/L;
C3         =  xp2/L;
D3         = -1/L;
    % Loop on TC
for iTC    =0.1:0.005:2.0;
indexTC=indexTC+1;
TC     =iTC;
bpole  = -1/TC;
TC_v(indexTC)=TC;
        % Setup the matrix coefficients
        %
A2vv = [((Iz2-Nrdot2)*Yv2)+(Nv2*Yrdot2)]/[((m2-Yvdot2)*(Iz2-Nrdot2))-
(Nvdot2*Yrdot2)];
A2vr = [((Yr2-(m2*u2))*(Iz2-Nrdot2))+(Nr2*Yrdot2)]/[((m2-Yvdot2)*(Iz2-
Nrdot2))-(Nvdot2*Yrdot2)];
B2v  = [-((Iz2-Nrdot2)*T)-(Yrdot2*T*xp2)]/[((m2-Yvdot2)*(Iz2-Nrdot2))-
(Nvdot2*Yrdot2)] ;
A2rv = [(Yv2*Nvdot2)+(Nv2*(m2-Yvdot2))]/[((m2-Yvdot2)*(Iz2-Nrdot2))-
(Nvdot2*Yrdot2)];
A2rr = [((Yr2-(m2*u2))*Nvdot2)+(Nr2*(m2-Yvdot2))]/[((m2-Yvdot2)*(Iz2-
Nrdot2))-(Nvdot2*Yrdot2)];
B2r  = [-(T*Nvdot2)-(T*xp2*(m2-Yvdot2))]/[((m2-Yvdot2)*(Iz2-Nrdot2))-
(Nvdot2*Yrdot2)];
A1vv = [((Iz1-Nrdot1)*Yv1)+(Nv1*Yrdot1)]/[((m1-Yvdot1)*(Iz1-Nrdot1))-
(Nvdot1*Yrdot1)];
A1vr = [(Yr1-m1*u1)*(Iz1-Nrdot1)+(Nr1*Yrdot1)]/[((m1-Yvdot1)*(Iz1-
Nrdot1))-(Nvdot1*Yrdot1)];
B1v  = [((Iz1-Nrdot1)*T)-(Yrdot1*T*xp1)]/[((m1-Yvdot1)*(Iz1-Nrdot1))-
(Nvdot1*Yrdot1)];
C1v  = [((Iz1-Nrdot1)*Ydel)+(Ndel*Yrdot1)]/[((m1-Yvdot1)*(Iz1-Nrdot1))-
(Nvdot1*Yrdot1)];
A1rv = [(Yv1*Nvdot1)+(Nv1*(m1-Yvdot1))]/[((m1-Yvdot1)*(Iz1-Nrdot1))-
(Nvdot1*Yrdot1)];
A1rr = [((Yr1-(m1*u1))*Nvdot1)+(Nr1*(m1-Yvdot1))]/[((m1-Yvdot1)*(Iz1-
Nrdot1))-(Nvdot1*Yrdot1)];
B1r  = [(T*Nvdot1)-(T*xp1*(m1-Yvdot1))]/[((m1-Yvdot1)*(Iz1-Nrdot1))-
(Nvdot1*Yrdot1)];
C1r  = [(Ydel*Nvdot1)+(Ndel*(m1-Yvdot1))]/[((m1-Yvdot1)*(Iz1-Nrdot1))-
(Nvdot1*Yrdot1)];
        % Find control gains
        %
C      = zeros(4);
C(1,3) = 1;
C(2,2) = A1vv;
C(2,3) = A1vr;
C(3,2) = A1rv;
C(3,3) = A1rr;
C(4,1) = 1;
C(4,2) = 1;
D      = zeros(4,1);
D(2,1) = C1v;
```

```
D(3,1) = C1r;
poles  = [bpole bpole-0.05 bpole-0.10 bpole-0.15];
k      = place(C,D,poles);
kpsi   = k(1,1);
kv     = k(1,2);
kr     = k(1,3);
ky     = k(1,4);
 %
% A matrix
%
A      = zeros(8);
A(1,1) = A2vv;
A(1,2) = A2vr;
A(1,5) = B2v/L;
A(1,6) = -B2v/L;
A(1,7) = B2v+((B2v*xp2)/L);
A(1,8) = (B2v*xp1)/L;
A(2,1) = A2rv;
A(2,2) = A2rr;
A(2,5) = B2r/L;
A(2,6) = -B2r/L;
A(2,7) = B2r+((B2r*xp2)/L);
A(2,8) = (B2r*xp1)/L;
A(3,3) = A1vv-(C1v*kv);
A(3,4) = A1vr-(C1v*kr);
A(3,5) = B1v/L;
A(3,6) = -B1v/L-(C1v*ky);
A(3,7) = (B1v*xp2)/L;
A(3,8) = B1v+((B1v*xp1)/L)-(C1v*kpsi);
A(4,3) = A1rv-(C1r*kv);
A(4,4) = A1rr-(C1r*kr);
A(4,5) = B1r/L;
A(4,6) = -B1r/L-(C1r*ky);
A(4,7) = (B1r*xp2)/L;
A(4,8) = B1r+((B1r*xp1)/L)-(C1r*kpsi);
A(5,1) = 1;
A(5,7) = 1;
A(6,3) = 1;
A(6,8) = 1;
A(7,2) = 1;
A(8,4) = 1;
%
% Find eigenvalues
%
B=eig(A);
F=real(B);
H_new=max(F);
%
% Detect if H changes its sign
%
if indexTC==1
H_old=H_new;
else
PROD=H_new*H_old;
if PROD > 0
```

```
%
% H does not change sign - keep going
%
H_old=H_new;
else
%
% H changed its sign - find critical TC by linear interpolation
%
index=index+1;
TC_crit(index)=-(H_old*TC_v(indexTC)-H_new*TC_v(indexTC-1))/(H_new-
H_old);
H_old=H_new;
%
bpole_crit  = -1/TC_crit(index);
poles  = [bpole_crit bpole_crit-0.05 bpole_crit-0.10 bpole_crit-0.15];
k      = place(C,D,poles);
kpsi   = k(1,1);
kv     = k(1,2);
kr     = k(1,3);
ky     = k(1,4);
A_crit      = zeros(8);
A_crit(1,1) = A2vv;
A_crit(1,2) = A2vr;
A_crit(1,5) = B2v/L;
A_crit(1,6) = -B2v/L;
A_crit(1,7) = B2v+((B2v*xp2)/L);
A_crit(1,8) = (B2v*xp1)/L;
A_crit(2,1) = A2rv;
A_crit(2,2) = A2rr;
A_crit(2,5) = B2r/L;
A_crit(2,6) = -B2r/L;
A_crit(2,7) = B2r+((B2r*xp2)/L);
A_crit(2,8) = (B2r*xp1)/L;
A_crit(3,3) = A1vv-(C1v*kv);
A_crit(3,4) = A1vr-(C1v*kr);
A_crit(3,5) = B1v/L;
A_crit(3,6) = -B1v/L-(C1v*ky);
A_crit(3,7) = (B1v*xp2)/L;
A_crit(3,8) = B1v+((B1v*xp1)/L)-(C1v*kpsi);
A_crit(4,3) = A1rv-(C1r*kv);
A_crit(4,4) = A1rr-(C1r*kr);
A_crit(4,5) = B1r/L;
A_crit(4,6) = -B1r/L-(C1r*ky);
A_crit(4,7) = (B1r*xp2)/L;
A_crit(4,8) = B1r+((B1r*xp1)/L)-(C1r*kpsi);
A_crit(5,1) = 1;
A_crit(5,7) = 1;
A_crit(6,3) = 1;
A_crit(6,8) = 1;
A_crit(7,2) = 1;
A_crit(8,4) = 1;
%
%Repeated coloumns are 4-5 and 6-7 of the Devcs
J=eig(A_crit);
[D_evcs,V_evls]=eig(A_crit);
```

```
T_matrix1(:,1)=D_evcs(:,1);
T_matrix1(:,2)=D_evcs(:,2);
T_matrix1(:,3)=D_evcs(:,3);
T_matrix1(:,4)=real(D_evcs(:,4));
T_matrix1(:,5)=imag(D_evcs(:,4));
T_matrix1(:,6)=real(D_evcs(:,6));
T_matrix1(:,7)=imag(D_evcs(:,6));
T_matrix1(:,8)=D_evcs(:,8);
%
T_matrix(:,1)=T_matrix1(:,6);
T_matrix(:,2)=T_matrix1(:,7);
T_matrix(:,3)=T_matrix1(:,1);
T_matrix(:,4)=T_matrix1(:,2);
T_matrix(:,5)=T_matrix1(:,3);
T_matrix(:,6)=T_matrix1(:,4);
T_matrix(:,7)=T_matrix1(:,5);
T_matrix(:,8)=T_matrix1(:,8);
%
m11=T_matrix(1,1);m12=T_matrix(1,2);
m21=T_matrix(2,1);m22=T_matrix(2,2);
m31=T_matrix(3,1);m32=T_matrix(3,2);
m41=T_matrix(4,1);m42=T_matrix(4,2);
m51=T_matrix(5,1);m52=T_matrix(5,2);
m81=T_matrix(8,1);m82=T_matrix(8,2);
m61=T_matrix(6,1);m62=T_matrix(6,2);
%
T_inverse=inv(T_matrix);
N11=T_inverse(1,1);
N12=T_inverse(1,2);
N13=T_inverse(1,3);
N14=T_inverse(1,4);
N15=T_inverse(1,5);
N11=T_inverse(1,1);
N21=T_inverse(2,1);
N22=T_inverse(2,2);
N23=T_inverse(2,3);
N24=T_inverse(2,4);
N25=T_inverse(2,5);
%
M=inv(T_matrix)*A_crit*T_matrix;

L11_1 = 1/6*B2v/L^3*m41^3-1/6*B2v/L^3*m81^3-1/(6*L^3)*B2v*m51^3*xp2^3;
L11_2 = -1/(2*L^2)*B2v*m51^3*xp2^2-1/6*B2v*m51^3-2/(3*L)*B2v*m51^3*xp2;
L11_3 = -1/(6*L)*B2v*m11^3*xp1-
/(6*L^3)*B2v*m11^3*xp1^3+1/(2*L^3)*B2v*m41*m81^2;
L11_4 =-1/(2*L^3)*B2v*m41^2*m81-1/L^2*B2v*m81*m51^2*xp2-
1/(2*L)*B2v*m81*m51^2;
L11_5= 1/(2*L^3)*B2v*m81*m51^2*xp2^2+1/L^2*B2v*m41*m51^2*xp2
       +1/(2*L)*B2v*m41*m51^2;
L11_6 =+1/(2*L^3)*B2v*m41*m51^2*xp2^2-1/(2*L^3)*B2v*xp1^2*m81*m11^2;
L11_7 = -1/(2*L^3)*B2v*xp1^2*m81*m11^2-1/(2*L^3)*B2v*xp1*m81^2*m11;
L11_8 = -1/(2*L^2)*B2v*m41^2*m51-1/(2*L^3)*B2v*m41^2*m51*xp2;
L11_9 =-1/(2*L^3)*B2v*xp1*m41^2*m11+1/(2*L^3)*B2v*xp1^2*m41*m11^2;
L11_10 =+1/L^3*B2v*xp2*m41*m81*m51+1/L^2*B2v*m41*m81*m51
        +1/L^3*B2v*m41*m81*m11*xp1;
```

```
L11_11 =-1/L^3*B2v*m81*m11*m51*xp2*xp1-1/L^2*B2v*m81*m11*m51*xp1;
L11_12 =+1/L^3*B2v*m41*m11*m51*xp2*xp1+1/L^2*B2v*m41*m11*m51*xp1;
L11_13 =-1/(2*L^3)*B2v*m11*m51^2*xp2^2*xp1-1/(2*L)*B2v*m11*m51^2*xp1;
L11_14 =-1/L^2*B2v*m11*m51^2*xp2*xp1-1/(2*L^3)*B2v*m11^2*m51*xp2*xp1^2;
L11_15 =-1/(2*L^2)*B2v*m11^2*m51*xp1^2;

L11 = L11_1+L11_2+L11_3+L11_4+L11_5+L11_6+L11_7+L11_8+L11_9+L11_10...
     +L11_11+L11_12+L11_13+L11_14+L11_15;

L12_1 = 1/6*B2v/L^3*m42^3-1/6*B2v/L^3*m82^3-1/(6*L^3)*B2v*m52^3*xp2^3;

L12_2 = -1/(2*L^2)*B2v*m52^3*xp2^2-1/6*B2v*m52^3-2/(3*L)*B2v*m52^3*xp2;
L12_3 = -1/(6*L)*B2v*m12^3*xp1-
1/(6*L^3)*B2v*m12^3*xp1^3+1/(2*L^3)*B2v*m42*m82^2;
L12_4 = -1/(2*L^3)*B2v*m42^2*m82-1/L^2*B2v*m82*m52^2*xp2-
1/(2*L)*B2v*m82*m52^2;
L12_5 = -1/(2*L^3)*B2v*m82*m52^2*xp2^2+1/L^2*B2v*m42*m52^2*xp2;
L12_6 = +1/(2*L)*B2v*m42*m52^2+1/(2*L^3)*B2v*m42*m52^2*xp2^2
L12_7 = -1/(2*L^3)*B2v*xp1^2*m82*m12^2-1/(2*L^3)*B2v*xp1^2*m82*m12^2;
L12_8 = -1/(2*L^3)*B2v*xp1*m82^2*m12-1/(2*L^2)*B2v*m42^2*m52;
L12_9 = -1/(2*L^3)*B2v*m42^2*m52*xp2-1/(2*L^3)*B2v*xp1*m42^2*m12;
L12_10 = +1/(2*L^3)*B2v*xp1^2*m42*m12^2+1/L^3*B2v*xp2*m42*m82*m52;
L12_11 = +1/L^2*B2v*m42*m82*m52+1/L^3*B2v*m42*m82*m12*xp1;
L12_12 = -1/L^3*B2v*m82*m12*m52*xp2*xp1-1/L^2*B2v*m82*m12*m52*xp1;
L12_13 = +1/L^3*B2v*m42*m12*m52*xp2*xp1+1/L^2*B2v*m42*m12*m52*xp1;
L12_14 = -1/(2*L^3)*B2v*m12*m52^2*xp2^2*xp1-1/(2*L)*B2v*m12*m52^2*xp1;
L12_15 = -1/L^2*B2v*m12*m52^2*xp2*xp1-1/(2*L^3)*B2v*m12^2*m52*xp2*xp1^2;
L12_16 = -1/(2*L^2)*B2v*m12^2*m52*xp1^2;

L12   = L12_1+L12_2+L12_3+L12_4+L12_5+L12_6+L12_7+L12_8+L12_9+L12_10...
          +L12_11+L12_12+L12_13+L12_14+L12_15+L12_16;

L13_1=1/2*B2v/L^3*m41^2*m42-1/2*B2v/L^3*m81^2*m82
-1/(2*L^3)*B2v*m51^2*m52*xp2^3;
L13_2=-3/(2*L^3)*B2v*m51^2*m52*xp2^2-1/2*B2v*m51^2*m52-
2/L*B2v*m51^2*m52*xp2;
L13_3 = -1/(2*L)*B2v*m11^2*m12*xp1-1/(2*L^3)*B2v*m11^2*m12*xp1^3;
L13_4=+1/L^3*B2v*m41*m81*m82+1/(2*L^3)*B2v*m42*m81^2-
1/(2*L^3)*B2v*m41^2*m82;
L13_5=-1/L^3*B2v*m41*m42*m81-2/(L^2)*B2v*m81*m51*m52*xp2-
1/L*B2v*m81*m51*m52;
L13_6 = -1/(L^3)*B2v*m81*m51*m52*xp2^2-1/(L^2)*B2v*m82*m51^2*xp2;
L13_7 = -1/(2*L)*B2v*m82*m51^2-1/(2*L^3)*B2v*m82*m51^2*xp2^2;
L13_8=2/(L^2)*B2v*m41*m51*m52*xp2+1/L*B2v*m41*m51*m52+1/L^3*B2v*m41*m51*
m52*xp2^2;
L13_9=1/L^2*B2v*m42*m51^2*xp2+1/(2*L)*B2v*m42*m51^2+1/(2*L^3)*B2v*m42*m5
1^2*xp2^2;
L13_10 = -1/L^3*B2v*xp1^2*m81*m11*m12-1/(2*L^3)*B2v*xp1^2*m82*m11^2;
L13_11 = -1/L^3*B2v*xp1^2*m81*m11*m12-1/(2*L^3)*B2v*xp1^2*m82*m11^2;
L13_12 = -1/(2*L^3)*B2v*xp1*m81^2*m12-1/L^3*B2v*xp1*m81*m82*m11;
L13_13=-1/(2*L^2)*B2v*m41^2*m52-1/(2*L^2)*B2v*m41^2*m52*xp2-
1/L^2*B2v*m41*m42*m51;
L13_14 = -1/L^3*B2v*m41*m42*m51*xp2-1/(2*L^3)*B2v*xp1*m41^2*m12;
L13_15 = -1/L^3*B2v*xp1*m41*m42*m11+1/L^3*B2v*xp1^2*m41*m11*m12;
L13_16 = +1/(2*L^3)*B2v*xp1^2*m42*m11^2+1/L^2*B2v*xp2*m41*m81*m52;
```

```
L13_17=1/L^2*B2v*m41*m81*m52+1/L^3*B2v*xp2*m41*m82*m51+1/L^2*B2v*m41*m82
*m51;
L13_18=+1/L^3*B2v*xp2*m42*m81*m51+1/L^2*B2v*m42*m81*m51+1/L^3*B2v*m41*m8
1*m12*xp1;
L13_19 =+1/L^3*B2v*m41*m82*m11*xp1+1/L^3*B2v*m42*m81*m11*xp1;
L13_20 = -1/L^3*B2v*m81*m11*m52*xp2*xp1-1/L^2*B2v*m81*m11*m52*xp1;
L13_21 = -1/L^3*B2v*m81*m12*m51*xp2*xp1-1/L^2*B2v*m81*m12*m51*xp1;
L13_22 = -1/L^3*B2v*m82*m11*m51*xp2*xp1-1/L^2*B2v*m82*m11*m51*xp1;
L13_23 = +1/L^3*B2v*m41*m11*m52*xp2*xp1+1/L^2*B2v*m41*m11*m52*xp1;
L13_24 = +1/L^3*B2v*m41*m12*m51*xp2*xp1+1/L^2*B2v*m41*m12*m51*xp1;
L13_25 = +1/L^3*B2v*m42*m11*m51*xp2*xp1+1/L^2*B2v*m42*m11*m51*xp1;
L13_26 = -1/L^3*B2v*m11*m51*m52*xp2^2*xp1-1/L*B2v*m11*m51*m52*xp1;
L13_27=-2/L^2*B2v*m11*m51*m52*xp2*xp1-1/(2*L^3)*B2v*m12*m51^2*xp2^2*xp1;
L13_28 = -1/(2*L)*B2v*m12*m51^2*xp1-1/L^2*B2v*m12*m51^2*xp2*xp1;
L13_29=-1/(2*L^3)*B2v*m11^2*m52*xp2*xp1^2-1/(2*L^2)*B2v*m11^2*m52*xp1^2;
L13_30 = -1/L^3*B2v*m11*m12*m51*xp2*xp1^2-1/L^2*B2v*m11*m12*m51*xp1^2;

L13   = L13_1+L13_2+L13_3+L13_4+L13_5+L13_6+L13_7+L13_8+L13_9+L13_10...
+L13_11+L13_12+L13_13+L13_14+L13_15+L13_16+L13_17+L13_18+L13_19Ö
+L13_20+L13_21+L13_22+L13_23+L13_24+L13_25+L13_26+L13_27+L13_28Ö
+L13_29+L13_30;

L14_1=1/2*B2v/L^3*m41*m42^2-1/2*B2v/L^3*m81*m82^2-
1/(2*L^3)*B2v*m51*m52^2*xp2^3;
L14_2=-3/(2*L^2)*B2v*m51*m52^2*xp2^2-1/2*B2v*m51*m52^2-
2/L*B2v*m51*m52^2*xp2;
L14_3=-1/(2*L)*B2v*m11*m12^2*xp1-1/(2*L^3)*B2v*m11*m12^2*xp1^3;
L14_4=+1/(2*L^3)*B2v*m41*m82^2+1/L^3*B2v*m42*m81*m82-
1/L^3*B2v*m41*m42*m82;
L14_5=-1/(2*L^3)*B2v*m42^2*m81-1/(L^2)*B2v*m81*m52^2*xp2-
1/(2*L)*B2v*m81*m52^2;
L14_6=-1/(2*L^3)*B2v*m81*m52^2*xp2^2-2/(L^2)*B2v*m82*m51*m52*xp2;
L14_7=-1/L*B2v*m82*m51*m52-
1/L^3*B2v*m82*m51*m52*xp2^2+1/L^2*B2v*m41*m52^2*xp2;
L14_8=+1/(2*L)*B2v*m41*m52^2*+1/(2*L^3)*B2v*m41*m52^2*xp2^2;
L14_9=+2/(L^2)*B2v*m42*m51*m52*xp2+1/L*B2v*m42*m51*m52+1/L^3*B2v*m42*m51
*m52*xp2^2;
L14_10=-1/(2*L^3)*B2v*xp1^2*m81*m12^2-1/L^3*B2v*xp1^2*m82*m11*m12;
L14_11=-1/(2*L^3)*B2v*xp1^2*m81*m12^2-1/L^3*B2v*xp1^2*m82*m11*m12;
L14_12=-1/L^3*B2v*xp1*m81*m82*m12-1/(2*L^3)*B2v*xp1*m82^2*m11;
L14_13=-1/L^2*B2v*m41*m42*m52-1/L^3*B2v*m41*m42*m52*xp2-
1/(2*L^2)*B2v*m42^2*m51;
L14_14=-1/(2*L^3)*B2v*m42^2*m51*xp2-1/L^3*B2v*xp1*m41*m42*m12;
L14_15=-1/(2*L^3)*B2v*xp1*m42^2*m11+1/(2*L^3)*B2v*xp1^2*m41*m12^2;
L14_16=+1/L^3*B2v*xp1^2*m42*m11*m12+1/L^3*B2v*xp2*m41*m82*m52+1/L^2*B2v*
m41*m82*m52;
L14_17=+1/L^3*B2v*xp2*m42*m81*m52+1/L^2*B2v*m42*m81*m52+1/L^3*B2v*xp2*m4
2*m82*m51;
L14_18=+1/L^2*B2v*m42*m82*m51+1/L^3*B2v*m41*m82*m12*xp1+1/L^3*B2v*m42*m8
1*m12*xp1;
L14_19=+1/L^3*B2v*m42*m82*m11*xp1-1/L^3*B2v*m81*m12*m52*xp2*xp1;
L14_20=-1/L^2*B2v*m81*m12*m52*xp1-1/L^3*B2v*m82*m11*m52*xp2*xp1;
L14_21=-1/L^2*B2v*m82*m11*m52*xp1-1/L^3*B2v*m82*m12*m51*xp2*xp1;
L14_22=-1/L^2*B2v*m82*m12*m51*xp1+1/L^3*B2v*m41*m12*m52*xp2*xp1;
L14_23=+1/L^2*B2v*m41*m12*m52*xp1+1/L^3*B2v*m42*m11*m52*xp2*xp1;
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L14_24=+1/L^2*B2v*m42*m11*m52*xp1+1/L^3*B2v*m42*m12*m51*xp2*xp1;
L14_25=+1/L^2*B2v*m42*m12*m51*xp1-1/(2*L^3)*B2v*m11*m52^2*xp2^2*xp1;
L14_26=-1/(2*L)*B2v*m11*m52^2*xp1-1/L^2*B2v*m11*m52^2*xp2*xp1;
L14_27=-1/(L^3)*B2v*m12*m51*m52*xp2^2*xp1-1/L*B2v*m12*m51*m52*xp1;
L14_28=-2/L^2*B2v*m12*m51*m52*xp2*xp1-1/(L^3)*B2v*m11*m12*m52*xp2*xp1^2;
L14_29=-1/(L^2)*B2v*m11*m12*m52*xp1^2-1/(2*L^3)*B2v*m12^2*m51*xp2*xp1^2;
L14_30= -1/L^2*B2v*m12^2*m51*xp1^2;

L14=L14_1+L14_2+L14_3+L14_4+L14_5+L14_6+L14_7+L14_8+L14_9+L14_10...

+L14_11+L14_12+L14_13+L14_14+L14_15+L14_16+L14_17+L14_18+L14_19+L14_20..

+L14_21+L14_22+L14_23+L14_24+L14_25+L14_26+L14_27+L14_28+L14_29+L14_30;

L21_1=1/(6*L^3)*B2r*m41^3-1/(6*L^3)*B2r*m81^3-1/(6*L^3)*B2r*m51^3*xp2^3;
L21_2=-1/(2*L^2)*B2r*m51^3*xp2^2-1/6*B2r*m51^3-2/(3*L)*B2r*m51^3*xp2;
L21_3=-1/(6*L)*B2r*m11^3*xp1-
1/(6*L^3)*B2r*m11^3*xp1^3+1/(2*L^3)*B2r*m41*m81^2;
L21_4=-1/(2*L^3)*B2r*m41^2*m81-1/(L^2)*B2r*m81*m51^2*xp2-
1/(2*L)*B2r*m81*m51^2;
L21_5=-1/(2*L^3)*B2r*m81*m51^2*xp2^2+1/(L^2)*B2r*m41*m51^2*xp2;
L21_6=+1/(2*L)*B2r*m41*m51^2+1/(2*L^3)*B2r*m41*m51^2*xp2^2;
L21_7=-1/(2*L^3)*B2r*xp1^2*m81*m11^2-1/(2*L^3)*B2r*xp1^2*m81*m11^2;
L21_8=-1/(2*L^3)*B2r*xp1*m81^2*m11-1/(2*L^2)*B2r*m41^2*m51;
L21_9=-1/(2*L^3)*B2r*m41^2*m51*xp2-1/(2*L^3)*B2r*xp1*m41^2*m11;
L21_10=+1/(2*L^3)*B2r*xp1^2*m41*m11^2+1/(L^3)*B2r*xp2*m41*m81*m51;
L21_11=+1/(L^2)*B2r*m41*m81*m51+1/(L^3)*B2r*m41*m81*m11*xp1;
L21_12=-1/(L^3)*B2r*m81*m11*m51*xp2*xp1-1/(L^2)*B2r*m81*m11*m51*xp1;
L21_13=+1/(L^3)*B2r*m41*m11*m51*xp2*xp1+1/(L^2)*B2r*m41*m11*m51*xp1;
L21_14=-1/(2*L^3)*B2r*m11*m51^2*xp2^2*xp1-1/(2*L)*B2r*m11*m51^2*xp1;
L21_15=-1/(L^2)*B2r*m11*m51^2*xp2*xp1-1/(2*L^3)*B2r*m11^2*m51*xp2*xp1^2;
L21_16=-1/(2*L^2)*B2r*m11^2*m51*xp1^2;

L21= L21_1+L21_2+L21_3+L21_4+L21_5+L21_6+L21_7+L21_8+L21_9+L21_10...
    +L21_11+L21_12+L21_13+L21_14+L21_15+L21_16;

L22_1=1/(6*L^3)*B2r*m42^3-1/(6*L^3)*B2r*m82^3-1/(6*L^3)*B2r*m52^3*xp2^3;
L22_2=-1/(2*L^2)*B2r*m52^3*xp2^2-1/6*B2r*m52^3-2/(3*L)*B2r*m52^3*xp2;
L22_3=-1/(6*L)*B2r*m12^3*xp1-
1/(6*L^3)*B2r*m12^3*xp1^3+1/(2*L^3)*B2r*m42*m82^2;
L22_4=-1/(2*L^3)*B2r*m42^2*m82-1/(L^2)*B2r*m82*m52^2*xp2-
1/(2*L)*B2r*m82*m52^2;
L22_5=-1/(2*L^3)*B2r*m82*m52^2*xp2^2+1/(L^2)*B2r*m42*m52^2*xp2;
L22_6=+1/(2*L)*B2r*m42*m52^2+1/(2*L^3)*B2r*m42*m52^2*xp2^2;
L22_7=-1/(2*L^3)*B2r*xp1^2*m82*m12^2-1/(2*L^3)*B2r*xp1^2*m82*m12^2;
L22_8=-1/(2*L^3)*B2r*xp1*m82^2*m12-1/(2*L^2)*B2r*m42^2*m52;
L22_9=-1/(2*L^3)*B2r*m42^2*m52*xp2-1/(2*L^3)*B2r*xp1*m42^2*m12;
L22_10=+1/(2*L^3)*B2r*xp1^2*m42*m12^2+1/(L^3)*B2r*xp2*m42*m82*m52;
L22_11=+1/(L^2)*B2r*m42*m82*m52+1/(L^3)*B2r*m42*m82*m12*xp1;
L22_12=-1/(L^3)*B2r*m82*m12*m52*xp2*xp1-1/(L^2)*B2r*m82*m12*m52*xp1;
L22_13= +1/(L^3)*B2r*m42*m12*m52*xp2*xp1+1/(L^2)*B2r*m42*m12*m52*xp1;
L22_14=-1/(2*L^3)*B2r*m12*m52^2*xp2^2*xp1-1/(2*L)*B2r*m12*m52^2*xp1;
L22_15 =-1/(L^2)*B2r*m12*m52^2*xp2*xp1-
1/(2*L^3)*B2r*m12^2*m52*xp2*xp1^2;
L22_16 = -1/(2*L^2)*B2r*m12^2*m52*xp1^2;
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L22 = L22_1+L22_2+L22_3+L22_4+L22_5+L22_6+L22_7+L22_8+L22_9+L22_10...
     +L22_11+L22_12+L22_13+L22_14+L22_15+L22_16;

L23_1=1/(2*L^3)*B2r*m41^2*m42-1/(2*L^3)*B2r*m81^2*m82-
1/(2*L^3)*B2r*m51^2*m52*xp2^3;
L23_2=-3/(2*L^2)*B2r*m51^2*m52*xp2^2-1/2*B2r*m51^2*m52-
2/L*B2r*m51^2*m52*xp2;
L23_3=-1/(2*L)*B2r*m11^2*m12*xp1-1/(2*L^3)*B2r*m11^2*m12*xp1^3;
L23_4=+1/(L^3)*B2r*m41*m81*m82+1/(2*L^3)*B2r*m42*m81^2-
1/(2*L^3)*B2r*m41^2*m82    ;
L23_5=-1/(L^3)*B2r*m41*m42*m81-2/(L^2)*B2r*m81*m51*m52*xp2-
1/L*B2r*m81*m51*m52;
L23_6=-1/(L^3)*B2r*m81*m51*m52*xp2^2-1/(2*L^2)*B2r*m82*m51^2*xp2;
L23_7=-1/(2*L)*B2r*m82*m51^2-1/(2*L^3)*B2r*m82*m51^2*xp2^2;
L23_8=+2/(L^2)*B2r*m41*m51*m52*xp2+1/L*B2r*m41*m51*m52;
L23_9=+1/(L^3)*B2r*m41*m51*m52*xp2^2+1/(L^2)*B2r*m42*m51^2*xp2;
L23_10=+1/(2*L)*B2r*m42*m51^2+1/(2*L^3)*B2r*m42*m51^2*xp2^2;
L23_11=-1/(L^3)*B2r*xp1^2*m81*m11*m12-1/(2*L^3)*B2r*xp1^2*m82*m11^2;
L23_12=-1/(L^3)*B2r*xp1^2*m81*m11*m12-1/(2*L^3)*B2r*xp1^2*m82*m11^2;
L23_13=-1/(2*L^3)*B2r*xp1*m81^2*m12-1/(L^3)*B2r*xp1*m81*m82*m11;
L23_14=-1/(2*L^2)*B2r*m41^2*m52-1/(2*L^3)*B2r*m41^2*m52*xp2;
L23_15=-1/(L^2)*B2r*m41*m42*m51-1/(L^3)*B2r*m41*m42*m51*xp2;
L23_16=-1/(2*L^3)*B2r*xp1*m41^2*m12-1/(L^3)*B2r*xp1*m41*m42*m11;
L23_17=+1/(L^3)*B2r*xp1^2*m41*m11*m12+1/(2*L^3)*B2r*xp1^2*m42*m11^2;
L23_18=+1/(L^3)*B2r*xp2*m41*m81*m52+1/(L^2)*B2r*m41*m81*m52;
L23_19=+1/(L^3)*B2r*xp2*m41*m82*m51+1/(L^2)*B2r*m41*m82*m51;
L23_20=+1/(L^3)*B2r*xp2*m42*m81*m51+1/(L^2)*B2r*m42*m81*m51;
L23_21=+1/(L^3)*B2r*m41*m81*m12*xp1+1/(L^3)*B2r*m41*m82*m11*xp1;
L23_22=+1/(L^3)*B2r*m42*m81*m11*xp1-1/(L^3)*B2r*m81*m11*m52*xp2*xp1;
L23_23=-1/(L^2)*B2r*m81*m11*m52*xp1-1/(L^3)*B2r*m81*m12*m51*xp2*xp1;
L23_24=-1/(L^2)*B2r*m81*m12*m51*xp1-1/(L^3)*B2r*m82*m11*m51*xp2*xp1;
L23_25=-1/(L^2)*B2r*m82*m11*m51*xp1+1/(L^3)*B2r*m41*m11*m52*xp2*xp1;
L23_26=+1/(L^2)*B2r*m41*m11*m52*xp1+1/(L^3)*B2r*m41*m12*m51*xp2*xp1;
L23_27=+1/(L^2)*B2r*m41*m12*m51*xp1+1/(L^3)*B2r*m42*m11*m51*xp2*xp1;
L23_28=+1/(L^2)*B2r*m42*m11*m51*xp1-1/(L^3)*B2r*m11*m51*m52*xp2^2*xp1;
L23_29=-1/L*B2r*m11*m51*m52*xp1-2/(L^2)*B2r*m11*m51*m52*xp2*xp1;
L23_30=-1/(2*L^3)*B2r*m12*m51^2*xp2^2*xp1-1/(2*L)*B2r*m12*m51^2*xp1;
L23_31=-1/(L^2)*B2r*m12*m51^2*xp2*xp1-1/(2*L^3)*B2r*m11^2*m52*xp2*xp1^2;
L23_32=-1/(2*L^2)*B2r*m11^2*m52*xp1^2-1/(L^3)*B2r*m11*m12*m51*xp2*xp1^2;
L23_33=-1/(L^2)*B2r*m11*m12*m51*xp1^2;


L23=L23_1+L23_2+L23_3+L23_4+L23_5+L23_6+L23_7+L23_8+L23_9+L23_10...

+L23_11+L23_12+L23_13+L23_14+L23_15+L23_16+L23_17+L23_18+L23_19+L23_20..

+L23_21+L23_22+L23_23+L23_24+L23_25+L23_26+L23_27+L23_28+L23_29+L23_30..
+L23_31+L23_32+L23_33;


L24_1=1/(2*L^3)*B2r*m41*m42^2-1/(2*L^3)*B2r*m81*m82^2-
1/(2*L^3)*B2r*m51*m52^2*xp2^3;
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L24_2=-3/(2*L^2)*B2r*m51*m52^2*xp2^2-1/2*B2r*m51*m52^2-
2/L*B2r*m51*m52^2*xp2;
L24_3 = -1/(2*L)*B2r*m11*m12^2*xp1-1/(2*L^3)*B2r*m11*m12^2*xp1^3;
L24_4=+1/(2*L^3)*B2r*m41*m82^2+1/(L^3)*B2r*m42*m81*m82-
1/(L^3)*B2r*m41*m42*m82;
L24_5=-1/(2*L^3)*B2r*m42^2*m81-1/(L^2)*B2r*m81*m52^2*xp2-
1/(2*L)*B2r*m81*m52^2;
L24_6=-1/(2*L^3)*B2r*m81*m52^2*xp2^2-1/(2*L^2)*B2r*m82*m51*m52*xp2;
L24_7=-1/L*B2r*m82*m51*m52-
1/(L^3)*B2r*m82*m51*m52*xp2^2+1/(L^2)*B2r*m41*m52^2*xp2;
L24_8=+1/(2*L)*B2r*m41*m52^2+1/(2*L^3)*B2r*m41*m52^2*xp2^2;
L24_9=+2/(L^2)*B2r*m42*m51*m52*xp2+1/L*B2r*m42*m51*m52+1/(L^3)*B2r*m42*m
51*m52*xp2^2;
L24_10=-1/(2*L^3)*B2r*xp1^2*m81*m12^2-1/(L^3)*B2r*xp1^2*m82*m11*m12;
L24_11=-1/(2*L^3)*B2r*xp1^2*m81*m12^2-1/(L^3)*B2r*xp1^2*m82*m11*m12;
L24_12=-1/(L^3)*B2r*xp1*m81*m82*m12-1/(2*L^3)*B2r*xp1*m82^2*m11;
L24_13=-1/(L^2)*B2r*m41*m42*m52-1/(L^3)*B2r*m41*m42*m52*xp2-
1/(2*L^2)*B2r*m42^2*m51;
L24_14=-1/(2*L^3)*B2r*m42^2*m51*xp2-1/(L^3)*B2r*xp1*m41*m42*m12;
L24_15=-1/(2*L^3)*B2r*xp1*m42^2*m11+1/(2*L^3)*B2r*xp1^2*m41*m12^2;
L24_16=+1/(L^3)*B2r*xp1^2*m42*m11*m12+1/(L^3)*B2r*xp2*m41*m82*m52;
L24_17=+1/(L^2)*B2r*m41*m82*m52+1/(L^3)*B2r*xp2*m42*m81*m52+1/(L^2)*B2r*
m42*m81*m52;
L24_18=+1/(L^3)*B2r*xp2*m42*m82*m51+1/(L^2)*B2r*m42*m82*m51;
L24_19=+1/(L^3)*B2r*m41*m82*m12*xp1+1/(L^3)*B2r*m42*m81*m12*xp1;
L24_20=+1/(L^3)*B2r*m42*m82*m11*xp1-1/(L^3)*B2r*m81*m12*m52*xp2*xp1;
L24_21=-1/(L^2)*B2r*m81*m12*m52*xp1-1/(L^3)*B2r*m82*m11*m52*xp2*xp1;
L24_22=-1/(L^2)*B2r*m82*m11*m52*xp1-1/(L^3)*B2r*m82*m12*m51*xp2*xp1;
L24_23=-1/(L^2)*B2r*m82*m12*m51*xp1+1/(L^3)*B2r*m41*m12*m52*xp2*xp1;
L24_24=+1/(L^2)*B2r*m41*m12*m52*xp1+1/(L^3)*B2r*m42*m11*m52*xp2*xp1;
L24_25=+1/(L^2)*B2r*m42*m11*m52*xp1+1/(L^3)*B2r*m42*m12*m51*xp2*xp1;
L24_26=+1/(L^2)*B2r*m42*m12*m51*xp1-1/(2*L^3)*B2r*m11*m52^2*xp2^2*xp1;
L24_27=-1/(2*L)*B2r*m11*m52^2*xp1-1/(L^2)*B2r*m11*m52^2*xp2*xp1;
L24_28=-1/(L^3)*B2r*m12*m51*m52*xp2^2*xp1-1/L*B2r*m12*m51*m52*xp1;
L24_29=-2/(L^2)*B2r*m12*m51*m52*xp2*xp1-
1/(L^3)*B2r*m11*m12*m52*xp2*xp1^2;
L24_30=-1/(L^2)*B2r*m11*m12*m52*xp1^2-1/(2*L^3)*B2r*m12^2*m51*xp2*xp1^2;
L24_31=-1/(2*L^2)*B2r*m12^2*m51*xp1^2;

L24= L24_1+L24_2+L24_3+L24_4+L24_5+L24_6+L24_7+L24_8+L24_9+L24_10...

+L24_11+L24_12+L24_13+L24_14+L24_15+L24_16+L24_17+L24_18+L24_19+L24_20..

+L24_21+L24_22+L24_23+L24_24+L24_25+L24_26+L24_27+L24_28+L24_29+L24_30..
+L24_31;

L31_1=1/(6*L^3)*B1v*m41^3-1/(6*L^3)*B1v*m81^3-1/(6*L)*B1v*m51^3*xp2;
L31_2=-1/(6*L^3)*B1v*m51^3*xp2^3-1/(2*L^2)*B1v*m11^3*xp1^2-
1/6*B1v*m11^3;
L31_3=-2/(3*L)*B1v*m11^3*xp1-
1/(6*L^3)*B1v*m11^3*xp1^3+1/(2*L^3)*B1v*m41*m81^2;
L31_4=+1/(2*L^3)*B1v*m41*m81^2-1/(2*L^3)*B1v*xp2^2*m81*m51^2;
L31_5=+1/(2*L^3)*B1v*xp2^2*m41*m51^2-1/(L^2)*B1v*m81*m11^2*xp1;
L31_6=-1/(2*L^3)*B1v*m81*m11^2*xp1^2-1/(2*L)*B1v*m81*m11^2;
L31_7=-1/(2*L^3)*B1v*xp2*m81^2*m51-1/(2*L^2)*B1v*m81^2*m11;
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L31_8=-1/(2*L^3)*B1v*m81^2*m11*xp1-1/(2*L^3)*B1v*xp2*m41^2*m51;
L31_9=-1/(2*L^2)*B1v*m41^2*m11-
1/(2*L^3)*B1v*m41^2*m11*xp1+1/(2*L)*B1v*m41*m11^2;
L31_10=+1/(L^2)*B1v*m41*m11^2*xp1+1/(2*L^3)*B1v*m41*m11^2*xp1^2;
L31_11=+1/(L^3)*B1v*xp2*m41*m81*m51+1/(L^2)*B1v*m41*m81*m11;
L31_12=+1/(L^3)*B1v*m41*m81*m11*xp1-1/(L^3)*B1v*m81*m11*m51*xp2*xp1;
L31_13=-1/(L^2)*B1v*m81*m11*m51*xp2+1/(L^3)*B1v*m41*m11*m51*xp2*xp1;
L31_14=+1/(L^2)*B1v*m41*m11*m51*xp2-1/(2*L^2)*B1v*m11*m51^2*xp2^2;
L31_15=-1/(2*L^3)*B1v*m11*m51^2*xp2^2*xp1-1/(2*L)*B1v*m11^2*m51*xp2;
L31_16=-1/(L^2)*B1v*m11^2*m51*xp2*xp1-1/(2*L^3)*B1v*m11^2*m51*xp2*xp1^2;
L31_17=-1/3*C1v/deltasat^2*kpsi^3*m11^3-1/3*C1v/deltasat^2*kv^3*m21^3;
L31_18=-1/3*C1v/deltasat^2*kr^3*m31^3-1/3*C1v/deltasat^2*ky^3*m41^3;
L31_19=-C1v/deltasat^2*kpsi^2*kv*m21*m11^2-
C1v/deltasat^2*kpsi^2*kr*m31*m11^2;
L31_20=-C1v/deltasat^2*kpsi^2*ky*m41*m11^2-
C1v/deltasat^2*kpsi*kv^2*m21^2*m11;
L31_21=-C1v/deltasat^2*kpsi*kr^2*m31^2*m11-
C1v/deltasat^2*kpsi*ky^2*m41^2*m11;
L31_22=-C1v/deltasat^2*kv^2*kr*m31*m21^2-
C1v/deltasat^2*kv^2*ky*m41*m21^2;
L31_23=-C1v/deltasat^2*kv*kr^2*m21*m31^2-
C1v/deltasat^2*kv*ky^2*m21*m41^2;
L31_24 = -C1v/deltasat^2*kr^2*ky*m41*m31^2-
C1v/deltasat^2*kr*ky^2*m31*m41^2;
L31_25 = -2*C1v/deltasat^2*kpsi*kv*kr*m31*m21*m11;
L31_26 = -2*C1v/deltasat^2*kpsi*kv*ky*m41*m21*m11;
L31_27 = -2*C1v/deltasat^2*kpsi*kr*ky*m41*m31*m11;
L31_28 = -2*C1v/deltasat^2*kv*kr*ky*m41*m21*m31;

L31   =  L31_1+L31_2+L31_3+L31_4+L31_5+L31_6+L31_7+L31_8+L31_9+L31_10...

+L31_11+L31_12+L31_13+L31_14+L31_15+L31_16+L31_17+L31_18+L31_19+L31_20..

+L31_21+L31_22+L31_23+L31_24+L31_25+L31_26+L31_27+L31_28;


L32_1=1/(6*L^3)*B1v*m42^3-1/(6*L^3)*B1v*m82^3-1/(6*L)*B1v*m52^3*xp2;
L32_2=-1/(6*L^3)*B1v*m52^3*xp2^3-1/(2*L^2)*B1v*m12^3*xp1^2-
1/6*B1v*m12^3;
L32_3=-2/(3*L)*B1v*m12^3*xp1-
1/(6*L^3)*B1v*m12^3*xp1^3+1/(2*L^3)*B1v*m42*m82^2;
L32_4=+1/(2*L^3)*B1v*m42*m82^2-1/(2*L^3)*B1v*xp2^2*m82*m52^2;
L32_5=+1/(2*L^3)*B1v*xp2^2*m42*m52^2-1/(L^2)*B1v*m82*m12^2*xp1;
L32_6=-1/(2*L^3)*B1v*m82*m12^2*xp1^2-1/(2*L)*B1v*m82*m12^2;
L32_7=-1/(2*L^3)*B1v*xp2*m82^2*m52-1/(2*L^2)*B1v*m82^2*m12;
L32_8=-1/(2*L^3)*B1v*m82^2*m12*xp1-1/(2*L^3)*B1v*xp2*m42^2*m52;
L32_9=-1/(2*L^2)*B1v*m42^2*m12-
1/(2*L^3)*B1v*m42^2*m12*xp1+1/(2*L)*B1v*m42*m12^2;
L32_10=+1/(L^2)*B1v*m42*m12^2*xp1+1/(2*L^3)*B1v*m42*m12^2*xp1^2;
L32_11=+1/(L^3)*B1v*xp2*m42*m82*m52+1/(L^2)*B1v*m42*m82*m12;
L32_12=+1/(L^3)*B1v*m42*m82*m12*xp1-1/(L^3)*B1v*m82*m12*m52*xp2*xp1;
L32_13=-1/(L^2)*B1v*m82*m12*m52*xp2+1/(L^3)*B1v*m42*m12*m52*xp2*xp1;
L32_14=+1/(L^2)*B1v*m42*m12*m52*xp2-1/(2*L^2)*B1v*m12*m52^2*xp2^2;
L32_15=-1/(2*L^3)*B1v*m12*m52^2*xp2^2*xp1-1/(2*L)*B1v*m12^2*m52*xp2;
L32_16=-1/(L^2)*B1v*m12^2*m52*xp2*xp1-1/(2*L^3)*B1v*m12^2*m52*xp2*xp1^2;
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L32_17=-1/3*C1v/deltasat^2*kpsi^3*m12^3-1/3*C1v/deltasat^2*kv^3*m22^3;
L32_18=-1/3*C1v/deltasat^2*kr^3*m32^3-1/3*C1v/deltasat^2*ky^3*m42^3;
L32_19=-C1v/deltasat^2*kpsi^2*kv*m22*m12^2-
C1v/deltasat^2*kpsi^2*kr*m32*m12^2;
L32_20=-C1v/deltasat^2*kpsi^2*ky*m42*m12^2-
C1v/deltasat^2*kpsi*kv^2*m22^2*m12;
L32_21=-C1v/deltasat^2*kpsi*kr^2*m32^2*m12-
C1v/deltasat^2*kpsi*ky^2*m42^2*m12;
L32_22=-C1v/deltasat^2*kv^2*kr*m32*m22^2-
C1v/deltasat^2*kv^2*ky*m42*m22^2;
L32_23=-C1v/deltasat^2*kv*kr^2*m22*m32^2-
C1v/deltasat^2*kv*ky^2*m22*m42^2;
L32_24=-C1v/deltasat^2*kr^2*ky*m42*m32^2-
C1v/deltasat^2*kr*ky^2*m32*m42^2;
L32_25 = -2*C1v/deltasat^2*kpsi*kv*kr*m32*m22*m12;
L32_26 = -2*C1v/deltasat^2*kpsi*kv*ky*m42*m22*m12;
L32_27 = -2*C1v/deltasat^2*kpsi*kr*ky*m42*m32*m12;
L32_28 = -2*C1v/deltasat^2*kv*kr*ky*m42*m22*m32;


L32   =  L32_1+L32_2+L32_3+L32_4+L32_5+L32_6+L32_7+L32_8+L32_9+L32_10...

+L32_11+L32_12+L32_13+L32_14+L32_15+L32_16+L32_17+L32_18+L32_19+L32_20..

+L32_21+L32_22+L32_23+L32_24+L32_25+L32_26+L32_27+L32_28;


L33_1=1/(2*L^3)*B1v*m41^2*m42-1/(2*L^3)*B1v*m81^2*m82-
1/(2*L)*B1v*m51^2*m52*xp2;
L33_2=-1/(2*L^3)*B1v*m51^2*m52*xp2^3-3/(2*L^2)*B1v*m11^2*m12*xp1^2;
L33_3=-1/2*B1v*m11^2*m12-2/L*B1v*m11^2*m12*xp1-
1/(2*L^3)*B1v*m11^2*m12*xp1^3;
L33_4=+1/(L^3)*B1v*m41*m81*m82+1/(2*L^3)*B1v*m42*m81^2+1/(L^3)*B1v*m41*m
81*m82 ;
L33_5=+1/(2*L^3)*B1v*m42*m81^2-1/(L^3)*B1v*xp2^2*m81*m51*m52;
L33_6=-1/(2*L^3)*B1v*xp2^2*m82*m51^2+1/(L^3)*B1v*xp2^2*m41*m51*m52;
L33_7=+1/(2*L^3)*B1v*xp2^2*m42*m51^2-2/(L^2)*B1v*m81*m11*m12*xp1;
L33_8=-1/(L^3)*B1v*m81*m11*m12*xp1^2-1/L*B1v*m81*m11*m12-
1/(L^2)*B1v*m82*m11^2*xp1 ;
L33_9=-1/(2*L^3)*B1v*m82*m11^2*xp1^2-1/(2*L)*B1v*m82*m11^2;
L33_10=-1/(2*L^3)*B1v*xp2*m81^2*m52-1/(L^3)*B1v*xp2*m81*m82*m51;
L33_11=-1/(2*L^2)*B1v*m81^2*m12-1/(2*L^3)*B1v*m81^2*m12*xp1;
L33_12=-1/(L^2)*B1v*m81*m82*m11-1/(L^3)*B1v*m81*m82*m11*xp1 ;
L33_13=-1/(2*L^3)*B1v*xp2*m41^2*m52-1/(L^3)*B1v*xp2*m41*m42*m51;
L33_14=-1/(2*L^2)*B1v*m41^2*m12-1/(2*L^3)*B1v*m41^2*m12*xp1-
1/(L^2)*B1v*m41*m42*m11;
L33_15=1/(L^3)*B1v*m41*m42*m11*xp1+1/L*B1v*m41*m11*m12+2/(L^2)*B1v*m41*m
11*m12*xp1;
L33_16=+1/(L^3)*B1v*m41*m11*m12*xp1^2+1/(2*L)*B1v*m42*m11^2;
L33_17=+1/(L^2)*B1v*m42*m11^2*xp1+1/(2*L^3)*B1v*m42*m11^2*xp1^2;
L33_18=+1/(L^3)*B1v*xp2*m41*m81*m52+1/(L^3)*B1v*xp2*m41*m82*m51;
L33_19=+1/(L^3)*B1v*xp2*m42*m81*m51+1/(L^2)*B1v*m41*m81*m12;
L33_20=+1/(L^3)*B1v*m41*m81*m12*xp1+1/(L^2)*B1v*m41*m82*m11;
L33_21=+1/(L^3)*B1v*m41*m82*m11*xp1+1/(L^2)*B1v*m42*m81*m11;
L33_22=+1/(L^3)*B1v*m42*m81*m11*xp1-1/(L^3)*B1v*m81*m11*m52*xp2*xp1;
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L33_23=-1/(L^2)*B1v*m81*m11*m52*xp2-1/(L^3)*B1v*m81*m12*m51*xp2*xp1;
L33_24 =-1/(L^2)*B1v*m81*m12*m51*xp2-1/(L^3)*B1v*m82*m11*m51*xp2*xp1;
L33_25=-1/(L^2)*B1v*m82*m11*m51*xp2+1/(L^3)*B1v*m41*m11*m52*xp2*xp1;
L33_26=+1/(L^2)*B1v*m41*m11*m52*xp2+1/(L^3)*B1v*m41*m12*m51*xp2*xp1;
L33_27=+1/(L^2)*B1v*m41*m12*m51*xp2+1/(L^3)*B1v*m42*m11*m51*xp2*xp1;
L33_28=+1/(L^2)*B1v*m42*m11*m51*xp2-1/(L^2)*B1v*m11*m51*m52*xp2^2 ;
L33_29=-1/(L^3)*B1v*m11*m51*m52*xp2^2*xp1-1/(2*L^2)*B1v*m12*m51^2*xp2^2;
L33_30=-1/(2*L^3)*B1v*m12*m51^2*xp2^2*xp1-1/(2*L)*B1v*m11^2*m52*xp2;
L33_31=-1/(L^2)*B1v*m11^2*m52*xp2*xp1-1/(2*L^3)*B1v*m11^2*m52*xp2*xp1^2;
L33_32=-1/L*B1v*m11*m12*m51*xp2-2/(L^2)*B1v*m11*m12*m51*xp2*xp1;
L33_33=-1/(L^3)*B1v*m11*m12*m51*xp2*xp1^2-
C1v/deltasat^2*kpsi^3*m11^2*m12;
L33_34=-C1v/deltasat^2*kv^3*m21^2*m22-C1v/deltasat^2*kr^3*m31^2*m32 ;
L33_35=-C1v/deltasat^2*ky^3*m41^2*m42-
2*C1v/deltasat^2*kpsi^2*kv*m21*m11*m12;
L33_36=-C1v/deltasat^2*kpsi^2*kv*m22*m11^2-
2*C1v/deltasat^2*kpsi^2*kr*m31*m11*m12;
L33_37=-C1v/deltasat^2*kpsi^2*kr*m32*m11^2-
2*C1v/deltasat^2*kpsi^2*ky*m41*m11*m12;
L33_38=-C1v/deltasat^2*kpsi^2*ky*m42*m11^2-
C1v/deltasat^2*kpsi*kv^2*m21^2*m12 ;
L33_39=-2*C1v/deltasat^2*kpsi*kv^2*m21*m22*m11-
C1v/deltasat^2*kpsi*kr^2*m31^2*m12;
L33_40=-2*C1v/deltasat^2*kpsi*kr^2*m31*m32*m11-
C1v/deltasat^2*kpsi*ky^2*m41^2*m12;
L33_41=-2*C1v/deltasat^2*kpsi*ky^2*m41*m42*m11-
2*C1v/deltasat^2*kv^2*kr*m31*m21*m22;
L33_42=-C1v/deltasat^2*kv^2*kr*m32*m21^2-
2*C1v/deltasat^2*kv^2*ky*m41*m21*m22;
L33_43=-C1v/deltasat^2*kv^2*ky*m42*m21^2-
2*C1v/deltasat^2*kv*kr^2*m21*m31*m32;
L33_44=-C1v/deltasat^2*kv*kr^2*m22*m31^2-
2*C1v/deltasat^2*kv*ky^2*m21*m41*m42;
L33_45=-C1v/deltasat^2*kv*ky^2*m22*m41^2-
2*C1v/deltasat^2*kr^2*ky*m41*m31*m32;
L33_46=-C1v/deltasat^2*kr^2*ky*m42*m31^2-
2*C1v/deltasat^2*kr*ky^2*m31*m41*m42;
L33_47=-C1v/deltasat^2*kr*ky^2*m32*m41^2-
2*C1v/deltasat^2*kpsi*kv*kr*m31*m21*m12;
L33_48=-2*C1v/deltasat^2*kpsi*kv*kr*m31*m22*m11;
L33_49=-2*C1v/deltasat^2*kpsi*kv*kr*m32*m21*m11;
L33_50=-2*C1v/deltasat^2*kpsi*kv*ky*m41*m21*m12;
L33_51=-2*C1v/deltasat^2*kpsi*kv*ky*m41*m22*m11;
L33_52=-2*C1v/deltasat^2*kpsi*kv*ky*m42*m21*m11;
L33_53=-2*C1v/deltasat^2*kpsi*kr*ky*m41*m31*m12;
L33_54=-2*C1v/deltasat^2*kpsi*kr*ky*m41*m32*m11;
L33_55=-2*C1v/deltasat^2*kpsi*kr*ky*m42*m31*m11;
L33_56=-2*C1v/deltasat^2*kv*kr*ky*m41*m21*m32-
2*C1v/deltasat^2*kv*kr*ky*m41*m22*m31;
L33_57=-2*C1v/deltasat^2*kv*kr*ky*m42*m21*m31;


L33   = L33_1+L33_2+L33_3+L33_4+L33_5+L33_6+L33_7+L33_8+L33_9+L33_10...

+L33_11+L33_12+L33_13+L33_14+L33_15+L33_16+L33_17+L33_18+L33_19+L33_20..
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+L33_21+L33_22+L33_23+L33_24+L33_25+L33_26+L33_27+L33_28+L33_29+L33_30..

+L33_31+L33_32+L33_33+L33_34+L33_35+L33_36+L33_37+L33_38+L33_39+L33_40..

+L33_41+L33_42+L33_44+L33_45+L33_46+L33_47+L33_48+L33_49+L33_50+L33_51..
+L33_52+L33_53+L33_54+L33_55+L33_56+L33_57;



L34_1=1/(2*L^3)*B1v*m41*m42^2-1/(2*L^3)*B1v*m81*m82^2-
1/(2*L)*B1v*m51*m52^2*xp2;
L34_2=-1/(2*L^3)*B1v*m51*m52^2*xp2^3-3/(2*L^2)*B1v*m11*m12^2*xp1^2;
L34_3=-1/2*B1v*m11*m12^2-2/L*B1v*m11*m12^2*xp1-
1/(2*L^3)*B1v*m11*m12^2*xp1^3;
L34_4=+1/(2*L^3)*B1v*m41*m82^2+1/(L^3)*B1v*m42*m81*m82+1/(2*L^3)*B1v*m41
*m82^2;
L34_5=+1/(L^3)*B1v*m42*m81*m82-1/(2*L^3)*B1v*xp2^2*m81*m52^2;
L34_6=-1/(L^3)*B1v*xp2^2*m82*m51*m52+1/(2*L^3)*B1v*xp2^2*m41*m52^2;
L34_7=+1/(L^3)*B1v*xp2^2*m42*m51*m52-1/(L^2)*B1v*m81*m12^2*xp1;
L34_8=-1/(2*L^3)*B1v*m81*m12^2*xp1^2-1/(2*L)*B1v*m81*m12^2;
L34_9=-2/(L^2)*B1v*m82*m11*m12*xp1-1/(L^3)*B1v*m82*m11*m12*xp1^2;
L34_10=-1/L*B1v*m82*m11*m12-1/(L^3)*B1v*xp2*m81*m82*m52-
1/(2*L^3)*B1v*xp2*m82^2*m51;
L34_11=-1/(L^2)*B1v*m81*m82*m12-1/(L^3)*B1v*m81*m82*m12*xp1-
1/(2*L^2)*B1v*m82^2*m11;
L34_12=-1/(2*L^3)*B1v*m82^2*m11*xp1-1/(L^3)*B1v*xp2*m41*m42*m52;
L34_13=-1/(2*L^3)*B1v*xp2*m42^2*m51-1/(L^2)*B1v*m41*m42*m12;
L34_14=-1/(L^3)*B1v*m41*m42*m12*xp1-1/(2*L^2)*B1v*m42^2*m11;
L34_15=-1/(2*L^3)*B1v*m42^2*m11*xp1+1/(2*L)*B1v*m41*m12^2+
1/(L^2)*B1v*m41*m12^2*xp1;
L34_16=+1/(2*L^3)*B1v*m41*m12^2*xp1^2+1/L*B1v*m42*m11*m12;
L34_17=+2/(L^2)*B1v*m42*m11*m12*xp1+1/(L^3)*B1v*m42*m11*m12*xp1^2;
L34_18=+1/(L^3)*B1v*xp2*m41*m82*m52+1/(L^3)*B1v*xp2*m42*m81*m52;
L34_19=+1/(L^3)*B1v*xp2*m42*m82*m51+1/(L^2)*B1v*m41*m82*m12;
L34_20=+1/(L^3)*B1v*m41*m82*m12*xp1+1/(L^2)*B1v*m42*m81*m12;
L34_21=+1/(L^3)*B1v*m42*m81*m12*xp1+1/(L^2)*B1v*m42*m82*m11;
L34_22=+1/(L^3)*B1v*m42*m82*m11*xp1-1/(L^3)*B1v*m81*m12*m52*xp2*xp1;
L34_23=-1/(L^2)*B1v*m81*m12*m52*xp2-1/(L^3)*B1v*m82*m11*m52*xp2*xp1;
L34_24=-1/(L^2)*B1v*m82*m11*m52*xp2-1/(L^3)*B1v*m82*m12*m51*xp2*xp1;
L34_25=-1/(L^2)*B1v*m82*m12*m51*xp2+1/(L^3)*B1v*m41*m12*m52*xp2*xp1;
L34_26= +1/(L^2)*B1v*m41*m12*m52*xp2+1/(L^3)*B1v*m42*m11*m52*xp2*xp1;
L34_27=+1/(L^2)*B1v*m42*m11*m52*xp2+1/(L^3)*B1v*m42*m12*m51*xp2*xp1;
L34_28=+1/(L^2)*B1v*m42*m12*m51*xp2-1/(2*L^2)*B1v*m11*m52^2*xp2^2;
L34_29=-1/(2*L^3)*B1v*m11*m52^2*xp2^2*xp1-1/(L^2)*B1v*m12*m51*m52*xp2^2;
L34_30=-1/(L^3)*B1v*m12*m51*m52*xp2^2*xp1-1/L*B1v*m11*m12*m52*xp2;
L34_31=-2/(L^2)*B1v*m11*m12*m52*xp2*xp1-
1/(L^3)*B1v*m11*m12*m52*xp2*xp1^2;
L34_32=-1/(2*L)*B1v*m12^2*m51*xp2-1/(L^2)*B1v*m12^2*m51*xp2*xp1;
L34_33=-1/(2*L^3)*B1v*m12^2*m51*xp2*xp1^2-
C1v/deltasat^2*kpsi^3*m11*m12^2;
L34_34=-C1v/deltasat^2*kv^3*m21*m22^2-C1v/deltasat^2*kr^3*m31*m32^2;
L34_35=-C1v/deltasat^2*ky^3*m41*m42^2-
C1v/deltasat^2*kpsi^2*kv*m21*m12^2;
```

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L34_36=-2*C1v/deltasat^2*kpsi^2*kv*m22*m11*m12-
C1v/deltasat^2*kpsi^2*kr*m31*m12^2;
L34_37=-2*C1v/deltasat^2*kpsi^2*kr*m32*m11*m12-
C1v/deltasat^2*kpsi^2*ky*m41*m12^2;
L34_38=-2*C1v/deltasat^2*kpsi^2*ky*m42*m11*m12;
L34_39=-2*C1v/deltasat^2*kpsi*kv^2*m21*m22*m12-
C1v/deltasat^2*kpsi*kv^2*m22^2*m11;
L34_40=-2*C1v/deltasat^2*kpsi*kr^2*m31*m32*m12-
C1v/deltasat^2*kpsi*kr^2*m32^2*m11;
L34_41=-2*C1v/deltasat^2*kpsi*ky^2*m41*m42*m12-
C1v/deltasat^2*kpsi*ky^2*m42^2*m11;
L34_42=-C1v/deltasat^2*kv^2*kr*m31*m22^2-
2*C1v/deltasat^2*kv^2*kr*m32*m21*m22;
L34_43=-C1v/deltasat^2*kv^2*ky*m41*m22^2-
2*C1v/deltasat^2*kv^2*ky*m42*m21*m22;
L34_44=-C1v/deltasat^2*kv*kr^2*m21*m32^2-
2*C1v/deltasat^2*kv*kr^2*m22*m31*m32;
L34_45 = -C1v/deltasat^2*kv*ky^2*m21*m42^2-
2*C1v/deltasat^2*kv*ky^2*m22*m41*m42;
L34_46 = -C1v/deltasat^2*kr^2*ky*m41*m32^2-
2*C1v/deltasat^2*kr^2*ky*m42*m31*m32;
L34_47 = -C1v/deltasat^2*kr*ky^2*m31*m42^2-
2*C1v/deltasat^2*kr*ky^2*m32*m41*m42;
L34_48 = -2*C1v/deltasat^2*kpsi*kv*kr*m31*m22*m12;
L34_49 = -2*C1v/deltasat^2*kpsi*kv*kr*m32*m21*m12;
L34_50 = -2*C1v/deltasat^2*kpsi*kv*kr*m32*m22*m11;
L34_51 = -2*C1v/deltasat^2*kpsi*kv*ky*m41*m22*m12;
L34_52 = -2*C1v/deltasat^2*kpsi*kv*ky*m42*m21*m12;
L34_53 = -2*C1v/deltasat^2*kpsi*kv*ky*m42*m22*m11;
L34_54 = -2*C1v/deltasat^2*kpsi*kr*ky*m41*m32*m12;
L34_55 = -2*C1v/deltasat^2*kpsi*kr*ky*m42*m31*m12;
L34_56 = -2*C1v/deltasat^2*kpsi*kr*ky*m42*m32*m11;
L34_57 = -2*C1v/deltasat^2*kv*kr*ky*m41*m22*m32-
2*C1v/deltasat^2*kv*kr*ky*m42*m21*m32;
L34_58 = -2*C1v/deltasat^2*kv*kr*ky*m42*m22*m31 ;


L34   = L34_1+L34_2+L34_3+L34_4+L34_5+L34_6+L34_7+L34_8+L34_9+L34_10...

+L34_11+L34_12+L34_13+L34_14+L34_15+L34_16+L34_17+L34_18+L34_19+L34_20..

+L34_21+L34_22+L34_23+L34_24+L34_25+L34_26+L34_27+L34_28+L34_29+L34_30..

+L34_31+L34_32+L34_33+L34_34+L34_35+L34_36+L34_37+L34_38+L34_39+L34_40..

+L34_41+L34_42+L34_44+L34_45+L34_46+L34_47+L34_48+L34_49+L34_50+L34_51..

+L34_52+L34_53+L34_54+L34_55+L34_56+L34_57+L34_58;


L41_1=1/(6*L^3)*B1r*m41^3-1/(6*L^3)*B1r*m81^3-1/(6*L)*B1r*m51^3*xp2;
L41_2=-1/(6*L^3)*B1r*m51^3*xp2^3-1/(2*L^2)*B1r*m11^3*xp1^2-
1/6*B1r*m11^3;
L41_3=-2/(3*L)*B1r*m11^3*xp1-
1/(6*L^3)*B1r*m11^3*xp1^3+1/(2*L^3)*B1r*m41*m81^2;
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L41_4=+1/(2*L^3)*B1r*m41*m81^2-1/(2*L^3)*B1r*xp2^2*m81*m51^2;
L41_5 = +1/(2*L^3)*B1r*xp2^2*m41*m51^2-1/(L^2)*B1r*m81*m11^2*xp1;
L41_6 = -1/(2*L^3)*B1r*m81*m11^2*xp1^2-1/(2*L)*B1r*m81*m11^2;
L41_7 = -1/(2*L^3)*B1r*xp2*m81^2*m51-1/(2*L^2)*B1r*m81^2*m11;
L41_8 = -1/(2*L^3)*B1r*m81^2*m11*xp1-1/(2*L^3)*B1r*xp2*m41^2*m51;
L41_9 = -1/(2*L^2)*B1r*m41^2*m11-
1/(2*L^3)*B1r*m41^2*m11*xp1+1/(2*L)*B1r*m41*m11^2;
L41_10 = +1/(L^2)*B1r*m41*m11^2*xp1+1/(2*L^3)*B1r*m41*m11^2*xp1^2;
L41_11 = +1/(L^3)*B1r*xp2*m41*m81*m51+1/(L^2)*B1r*m41*m81*m11;
L41_12 = +1/(L^3)*B1r*m41*m81*m11*xp1-1/(L^3)*B1r*m81*m11*m51*xp2*xp1;
L41_13 = -1/(L^2)*B1r*m81*m11*m51*xp2+1/(L^3)*B1r*m41*m11*m51*xp2*xp1;
L41_14 = +1/(L^2)*B1r*m41*m11*m51*xp2-1/(2*L^2)*B1r*m11*m51^2*xp2^2;
L41_15 = -1/(2*L^3)*B1r*m11*m51^2*xp2^2*xp1-1/(2*L)*B1r*m11^2*m51*xp2;
L41_16 = -1/(L^2)*B1r*m11^2*m51*xp2*xp1-
1/(2*L^3)*B1r*m11^2*m51*xp2*xp1^2;
L41_17 = -1/3*C1r/deltasat^2*kpsi^3*m11^3-1/3*C1r/deltasat^2*kv^3*m21^3;
L41_18 = -1/3*C1r/deltasat^2*kr^3*m31^3-1/3*C1r/deltasat^2*ky^3*m41^3;
L41_19 = -C1r/deltasat^2*kpsi^2*kv*m21*m11^2-
C1r/deltasat^2*kpsi^2*kr*m31*m11^2;
L41_20 = -C1r/deltasat^2*kpsi^2*ky*m41*m11^2-
C1r/deltasat^2*kpsi*kv^2*m21^2*m11;
L41_21 = -C1r/deltasat^2*kpsi*kr^2*m31^2*m11-
C1r/deltasat^2*kpsi*ky^2*m41^2*m11;
L41_22 = -C1r/deltasat^2*kv^2*kr*m31*m21^2-
C1r/deltasat^2*kv^2*ky*m41*m21^2;
L41_23 = -C1r/deltasat^2*kv*kr^2*m21*m31^2-
C1r/deltasat^2*kv*ky^2*m21*m41^2;
L41_24 = -C1r/deltasat^2*kr^2*ky*m41*m31^2-
C1r/deltasat^2*kr*ky^2*m31*m41^2;
L41_25 = -2*C1r/deltasat^2*kpsi*kv*kr*m31*m21*m11;
L41_26 = -2*C1r/deltasat^2*kpsi*kv*ky*m41*m21*m11;
L41_27 = -2*C1r/deltasat^2*kpsi*kr*ky*m41*m31*m11;
L41_28 = -2*C1r/deltasat^2*kv*kr*ky*m41*m21*m31;



L41=L41_1+L41_2+L41_3+L41_4+L41_5+L41_6+L41_7+L41_8+L41_9+L41_10...

+L41_11+L41_12+L41_13+L41_14+L41_15+L41_16+L41_17+L41_18+L41_19+L41_20..

+L41_21+L41_22+L41_23+L41_24+L41_25+L41_26+L41_27+L41_28;

L42_1=1/(6*L^3)*B1r*m42^3-1/(6*L^3)*B1r*m82^3-1/(6*L)*B1r*m52^3*xp2 ;
L42_2=-1/(6*L^3)*B1r*m52^3*xp2^3-1/(2*L^2)*B1r*m12^3*xp1^2-1/6*B1r*m12^3
L42_3=-2/(3*L)*B1r*m12^3*xp1-
1/(6*L^3)*B1r*m12^3*xp1^3+1/(2*L^3)*B1r*m42*m82^2;
L42_4=+1/(2*L^3)*B1r*m42*m82^2-1/(2*L^3)*B1r*xp2^2*m82*m52^2 ;
L42_5=+1/(2*L^3)*B1r*xp2^2*m42*m52^2-1/(L^2)*B1r*m82*m12^2*xp1;
L42_6=-1/(2*L^3)*B1r*m82*m12^2*xp1^2-1/(2*L)*B1r*m82*m12^2    ;
L42_7=-1/(2*L^3)*B1r*xp2*m82^2*m52-1/(2*L^2)*B1r*m82^2*m12;
L42_8=-1/(2*L^3)*B1r*m82^2*m12*xp1-1/(2*L^3)*B1r*xp2*m42^2*m52;
L42_9=-1/(2*L^2)*B1r*m42^2*m12-
1/(2*L^3)*B1r*m42^2*m12*xp1+1/(2*L)*B1r*m42*m12^2;
L42_10=+1/(L^2)*B1r*m42*m12^2*xp1+1/(2*L^3)*B1r*m42*m12^2*xp1^2      ;
L42_11 = +1/(L^3)*B1r*xp2*m42*m82*m52+1/(L^2)*B1r*m42*m82*m12;
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L42_12 =  +1/(L^3)*B1r*m42*m82*m12*xp1-1/(L^3)*B1r*m82*m12*m52*xp2*xp1;
L42_13 = -1/(L^2)*B1r*m82*m12*m52*xp2+1/(L^3)*B1r*m42*m12*m52*xp2*xp1;
L42_14=+1/(L^2)*B1r*m42*m12*m52*xp2-1/(2*L^2)*B1r*m12*m52^2*xp2^2  ;
L42_15=-1/(2*L^3)*B1r*m12*m52^2*xp2^2*xp1-1/(2*L)*B1r*m12^2*m52*xp2;
L42_16=-1/(L^2)*B1r*m12^2*m52*xp2*xp1-1/(2*L^3)*B1r*m12^2*m52*xp2*xp1^2;
L42_17=-1/3*C1r/deltasat^2*kpsi^3*m12^3-1/3*C1r/deltasat^2*kv^3*m22^3;
L42_18 =-1/3*C1r/deltasat^2*kr^3*m32^3-1/3*C1r/deltasat^2*ky^3*m42^3   ;
L42_19 = -C1r/deltasat^2*kpsi^2*kv*m22*m12^2-
C1r/deltasat^2*kpsi^2*kr*m32*m12^2;
L42_20 =  -C1r/deltasat^2*kpsi^2*ky*m42*m12^2-
C1r/deltasat^2*kpsi*kv^2*m22^2*m12;
L42_21 = -C1r/deltasat^2*kpsi*kr^2*m32^2*m12-
C1r/deltasat^2*kpsi*ky^2*m42^2*m12;
L42_22 = -C1r/deltasat^2*kv^2*kr*m32*m22^2-
C1r/deltasat^2*kv^2*ky*m42*m22^2     ;
L42_23 = -C1r/deltasat^2*kv*kr^2*m22*m32^2-
C1r/deltasat^2*kv*ky^2*m22*m42^2;
L42_24 =-C1r/deltasat^2*kr^2*ky*m42*m32^2-
C1r/deltasat^2*kr*ky^2*m32*m42^2;
L42_25 = -2*C1r/deltasat^2*kpsi*kv*kr*m32*m22*m12;
L42_26 = -2*C1r/deltasat^2*kpsi*kv*ky*m42*m22*m12;
L42_27 = -2*C1r/deltasat^2*kpsi*kr*ky*m42*m32*m12;
L42_28 = -2*C1r/deltasat^2*kv*kr*ky*m42*m22*m32;


L42=L42_1+L42_2+L42_3+L42_4+L42_5+L42_6+L42_7+L42_8+L42_9+L42_10...

+L42_11+L42_12+L42_13+L42_14+L42_15+L42_16+L42_17+L42_18+L42_19+L42_20..

+L42_21+L42_22+L42_23+L42_24+L42_25+L42_26+L42_27+L42_28;


L43_1=1/(2*L^3)*B1r*m41^2*m42-1/(2*L^3)*B1r*m81^2*m82-
1/(2*L)*B1r*m51^2*m52*xp2;
L43_2=-1/(2*L^3)*B1r*m51^2*m52*xp2^3-3/(2*L^2)*B1r*m11^2*m12*xp1^2   ;
L43_3=-1/2*B1r*m11^2*m12-2/L*B1r*m11^2*m12*xp1-
1/(2*L^3)*B1r*m11^2*m12*xp1^3;
L43_4=+1/(L^3)*B1r*m41*m81*m82+1/(2*L^3)*B1r*m42*m81^2+1/(L^3)*B1r*m41*m
81*m82    ;
L43_5=+1/(2*L^3)*B1r*m42*m81^2-1/(L^3)*B1r*xp2^2*m81*m51*m52;
L43_6=-1/(2*L^3)*B1r*xp2^2*m82*m51^2+1/(L^3)*B1r*xp2^2*m41*m51*m52     ;
L43_7=+1/(2*L^3)*B1r*xp2^2*m42*m51^2-2/(L^2)*B1r*m81*m11*m12*xp1;
L43_8=-1/(L^3)*B1r*m81*m11*m12*xp1^2-1/L*B1r*m81*m11*m12-
1/(L^2)*B1r*m82*m11^2*xp1    ;
L43_9=-1/(2*L^3)*B1r*m82*m11^2*xp1^2-1/(2*L)*B1r*m82*m11^2;
L43_10=-1/(2*L^3)*B1r*xp2*m81^2*m52-1/(L^3)*B1r*xp2*m81*m82*m51    ;
L43_11=-1/(2*L^2)*B1r*m81^2*m12-1/(2*L^3)*B1r*m81^2*m12*xp1-
1/(L^2)*B1r*m81*m82*m11;
L43_12=-1/(L^3)*B1r*m81*m82*m11*xp1-1/(2*L^3)*B1r*xp2*m41^2*m52    ;
L43_13=-1/(L^3)*B1r*xp2*m41*m42*m51-1/(2*L^2)*B1r*m41^2*m12;
L43_14=-1/(2*L^3)*B1r*m41^2*m12*xp1-1/(L^2)*B1r*m41*m42*m11 ;
L43_15=1/(L^3)*B1r*m41*m42*m11*xp1+1/L*B1r*m41*m11*m12+2/(L^2)*B1r*m41*m
11*m12*xp1;
L43_16=+1/(L^3)*B1r*m41*m11*m12*xp1^2+1/(2*L)*B1r*m42*m11^2    ;
L43_17=+1/(L^2)*B1r*m42*m11^2*xp1+1/(2*L^3)*B1r*m42*m11^2*xp1^2;
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L43_18=+1/(L^3)*B1r*xp2*m41*m81*m52+1/(L^3)*B1r*xp2*m41*m82*m51 ;
L43_19=+1/(L^3)*B1r*xp2*m42*m81*m51+1/(L^2)*B1r*m41*m81*m12;
L43_20=+1/(L^3)*B1r*m41*m81*m12*xp1+1/(L^2)*B1r*m41*m82*m11 ;
L43_21=+1/(L^3)*B1r*m41*m82*m11*xp1+1/(L^2)*B1r*m42*m81*m11;
L43_22=+1/(L^3)*B1r*m42*m81*m11*xp1-1/(L^3)*B1r*m81*m11*m52*xp2*xp1    ;
L43_23=-1/(L^2)*B1r*m81*m11*m52*xp2-1/(L^3)*B1r*m81*m12*m51*xp2*xp1;
L43_24=-1/(L^2)*B1r*m81*m12*m51*xp2-1/(L^3)*B1r*m82*m11*m51*xp2*xp1 ;
L43_25=-1/(L^2)*B1r*m82*m11*m51*xp2+1/(L^3)*B1r*m41*m11*m52*xp2*xp1;
L43_26=+1/(L^2)*B1r*m41*m11*m52*xp2+1/(L^3)*B1r*m41*m12*m51*xp2*xp1;
L43_27=+1/(L^2)*B1r*m41*m12*m51*xp2+1/(L^3)*B1r*m42*m11*m51*xp2*xp1;
L43_28=+1/(L^2)*B1r*m42*m11*m51*xp2-1/(L^2)*B1r*m11*m51*m52*xp2^2   ;
L43_29=-1/(L^3)*B1r*m11*m51*m52*xp2^2*xp1-1/(2*L^2)*B1r*m12*m51^2*xp2^2;
L43_30=-1/(2*L^3)*B1r*m12*m51^2*xp2^2*xp1-1/(2*L)*B1r*m11^2*m52*xp2    ;
L43_31=-1/(L^2)*B1r*m11^2*m52*xp2*xp1-1/(2*L^3)*B1r*m11^2*m52*xp2*xp1^2;
L43_32=-1/L*B1r*m11*m12*m51*xp2-2/(L^2)*B1r*m11*m12*m51*xp2*xp1     ;
L43_33=-1/(L^3)*B1r*m11*m12*m51*xp2*xp1^2-
C1r/deltasat^2*kpsi^3*m11^2*m12;
L43_34=-C1r/deltasat^2*kv^3*m21^2*m22-C1r/deltasat^2*kr^3*m31^2*m32;
L43_35=-C1r/deltasat^2*ky^3*m41^2*m42-
2*C1r/deltasat^2*kpsi^2*kv*m21*m11*m12;
L43_36=-C1r/deltasat^2*kpsi^2*kv*m22*m11^2-
2*C1r/deltasat^2*kpsi^2*kr*m31*m11*m12    ;
L43_37=-C1r/deltasat^2*kpsi^2*kr*m32*m11^2-
2*C1r/deltasat^2*kpsi^2*ky*m41*m11*m12;
L43_38=-C1r/deltasat^2*kpsi^2*ky*m42*m11^2-
C1r/deltasat^2*kpsi*kv^2*m21^2*m12    ;
L43_39 = -2*C1r/deltasat^2*kpsi*kv^2*m21*m22*m11-
C1r/deltasat^2*kpsi*kr^2*m31^2*m12;
L43_40 =  -2*C1r/deltasat^2*kpsi*kr^2*m31*m32*m11-
C1r/deltasat^2*kpsi*ky^2*m41^2*m12 ;
L43_41 =-2*C1r/deltasat^2*kpsi*ky^2*m41*m42*m11-
2*C1r/deltasat^2*kv^2*kr*m31*m21*m22 ;
L43_42 = -C1r/deltasat^2*kv^2*kr*m32*m21^2-
2*C1r/deltasat^2*kv^2*ky*m41*m21*m22     ;
L43_43 = -C1r/deltasat^2*kv^2*ky*m42*m21^2-
2*C1r/deltasat^2*kv*kr^2*m21*m31*m32;
L43_44 =  -C1r/deltasat^2*kv*kr^2*m22*m31^2-
2*C1r/deltasat^2*kv*ky^2*m21*m41*m42;
L43_45 = -C1r/deltasat^2*kv*ky^2*m22*m41^2-
2*C1r/deltasat^2*kr^2*ky*m41*m31*m32;
L43_46 =  -C1r/deltasat^2*kr^2*ky*m42*m31^2-
2*C1r/deltasat^2*kr*ky^2*m31*m41*m42;
L43_47 = -C1r/deltasat^2*kr*ky^2*m32*m41^2-
2*C1r/deltasat^2*kpsi*kv*kr*m31*m21*m12;
L43_48 =  -2*C1r/deltasat^2*kpsi*kv*kr*m31*m22*m11    ;
L43_49 = -2*C1r/deltasat^2*kpsi*kv*kr*m32*m21*m11;
L43_50 = -2*C1r/deltasat^2*kpsi*kv*ky*m41*m21*m12 ;
L43_51 = -2*C1r/deltasat^2*kpsi*kv*ky*m41*m22*m11;
L43_52 =  -2*C1r/deltasat^2*kpsi*kv*ky*m42*m21*m11;
L43_53 = -2*C1r/deltasat^2*kpsi*kr*ky*m41*m31*m12;
L43_54 = -2*C1r/deltasat^2*kpsi*kr*ky*m41*m32*m11 ;
L43_55 = -2*C1r/deltasat^2*kpsi*kr*ky*m42*m31*m11;
L43_56 = -2*C1r/deltasat^2*kv*kr*ky*m41*m21*m32-
2*C1r/deltasat^2*kv*kr*ky*m41*m22*m31     ;
L43_57 = -2*C1r/deltasat^2*kv*kr*ky*m42*m21*m31;
```

```
L43= L43_1+L43_2+L43_3+L43_4+L43_5+L43_6+L43_7+L43_8+L43_9+L43_10...

+L43_11+L43_12+L43_13+L43_14+L43_15+L43_16+L43_17+L43_18+L43_19+L43_20..

+L43_21+L43_22+L43_23+L43_24+L43_25+L43_26+L43_27+L43_28+L43_29+L43_30..

+L43_31+L43_32+L43_33+L43_34+L43_35+L43_36+L43_37+L43_38+L43_39+L43_40..

+L43_41+L43_42+L43_44+L43_45+L43_46+L43_47+L43_48+L43_49+L43_50+L43_51..
L43_52+L43_53+L43_54+L43_55+L43_56+L43_57;


L44_1=1/(2*L^3)*B1r*m41*m42^2-1/(2*L^3)*B1r*m81*m82^2-
1/(2*L)*B1r*m51*m52^2*xp2;
L44_2=-1/(2*L^3)*B1r*m51*m52^2*xp2^3-3/(2*L^2)*B1r*m11*m12^2*xp1^2;
L44_3=-1/2*B1r*m11*m12^2-2/L*B1r*m11*m12^2*xp1-
1/(2*L^3)*B1r*m11*m12^2*xp1^3;
L44_4=+1/(2*L^3)*B1r*m41*m82^2+1/(L^3)*B1r*m42*m81*m82+1/(2*L^3)*B1r*m41
*m82^2;
L44_5=+1/(L^3)*B1r*m42*m81*m82-1/(2*L^3)*B1r*xp2^2*m81*m52^2;
L44_6=-1/(L^3)*B1r*xp2^2*m82*m51*m52+1/(2*L^3)*B1r*xp2^2*m41*m52^2;
L44_7=+1/(L^3)*B1r*xp2^2*m42*m51*m52-1/(L^2)*B1r*m81*m12^2*xp1;
L44_8=-1/(2*L^3)*B1r*m81*m12^2*xp1^2-1/(2*L)*B1r*m81*m12^2;
L44_9=-2/(L^2)*B1r*m82*m11*m12*xp1-1/(L^3)*B1r*m82*m11*m12*xp1^2;
L44_10=-1/L*B1r*m82*m11*m12-1/(L^3)*B1r*xp2*m81*m82*m52-
1/(2*L^3)*B1r*xp2*m82^2*m51;
L44_11=-1/(L^2)*B1r*m81*m82*m12-1/(L^3)*B1r*m81*m82*m12*xp1-
1/(2*L^2)*B1r*m82^2*m11 ;
L44_12=-1/(2*L^3)*B1r*m82^2*m11*xp1-1/(L^3)*B1r*xp2*m41*m42*m52;
L44_13=-1/(2*L^3)*B1r*xp2*m42^2*m51-1/(L^2)*B1r*m41*m42*m12;
L44_14=-1/(L^3)*B1r*m41*m42*m12*xp1-1/(2*L^2)*B1r*m42^2*m11;
L44_15=-1/(2*L^3)*B1r*m42^2*m11*xp1+1/(2*L)*B1r*m41*m12^2+
1/(L^2)*B1r*m41*m12^2*xp1;
L44_16=+1/(2*L^3)*B1r*m41*m12^2*xp1^2+1/L*B1r*m42*m11*m12;
L44_17=+2/(L^2)*B1r*m42*m11*m12*xp1+1/(L^3)*B1r*m42*m11*m12*xp1^2;
L44_18=+1/(L^3)*B1r*xp2*m41*m82*m52+1/(L^3)*B1r*xp2*m42*m81*m52;
L44_19=+1/(L^3)*B1r*xp2*m42*m82*m51+1/(L^2)*B1r*m41*m82*m12;
L44_20=+1/(L^3)*B1r*m41*m82*m12*xp1+1/(L^2)*B1r*m42*m81*m12;
L44_21=+1/(L^3)*B1r*m42*m81*m12*xp1+1/(L^2)*B1r*m42*m82*m11;
L44_22=+1/(L^3)*B1r*m42*m82*m11*xp1-1/(L^3)*B1r*m81*m12*m52*xp2*xp1;
L44_23=-1/(L^2)*B1r*m81*m12*m52*xp2-1/(L^3)*B1r*m82*m11*m52*xp2*xp1;
L44_24=-1/(L^2)*B1r*m82*m11*m52*xp2-1/(L^3)*B1r*m82*m12*m51*xp2*xp1;
L44_25=-1/(L^2)*B1r*m82*m12*m51*xp2+1/(L^3)*B1r*m41*m12*m52*xp2*xp1;
L44_26=+1/(L^2)*B1r*m41*m12*m52*xp2+1/(L^3)*B1r*m42*m11*m52*xp2*xp1;
L44_27=+1/(L^2)*B1r*m42*m11*m52*xp2+1/(L^3)*B1r*m42*m12*m51*xp2*xp1;
L44_28=+1/(L^2)*B1r*m42*m12*m51*xp2-1/(2*L^2)*B1r*m11*m52^2*xp2^2;
L44_29=-1/(2*L^3)*B1r*m11*m52^2*xp2^2*xp1-1/(L^2)*B1r*m12*m51*m52*xp2^2;
L44_30=-1/(L^3)*B1r*m12*m51*m52*xp2^2*xp1-1/L*B1r*m11*m12*m52*xp2;
L44_31=-2/(L^2)*B1r*m11*m12*m52*xp2*xp1-
1/(L^3)*B1r*m11*m12*m52*xp2*xp1^2;
L44_32=-1/(2*L)*B1r*m12^2*m51*xp2-1/(L^2)*B1r*m12^2*m51*xp2*xp1;
L44_33=-1/(2*L^3)*B1r*m12^2*m51*xp2*xp1^2-
C1r/deltasat^2*kpsi^3*m11*m12^2;
```

```
L44_34 =-C1r/deltasat^2*kv^3*m21*m22^2-C1r/deltasat^2*kr^3*m31*m32^2;
L44_35=-C1r/deltasat^2*ky^3*m41*m42^2-
C1r/deltasat^2*kpsi^2*kv*m21*m12^2;
L44_36=-2*C1r/deltasat^2*kpsi^2*kv*m22*m11*m12-
C1r/deltasat^2*kpsi^2*kr*m31*m12^2;
L44_37=-2*C1r/deltasat^2*kpsi^2*kr*m32*m11*m12-
C1r/deltasat^2*kpsi^2*ky*m41*m12^2;
L44_38=-2*C1r/deltasat^2*kpsi^2*ky*m42*m11*m12;
L44_39=-2*C1r/deltasat^2*kpsi*kv^2*m21*m22*m12-
C1r/deltasat^2*kpsi*kv^2*m22^2*m11;
L44_40 = -2*C1r/deltasat^2*kpsi*kr^2*m31*m32*m12-
C1r/deltasat^2*kpsi*kr^2*m32^2*m11;
L44_41=-2*C1r/deltasat^2*kpsi*ky^2*m41*m42*m12-
C1r/deltasat^2*kpsi*ky^2*m42^2*m11;
L44_42=-C1r/deltasat^2*kv^2*kr*m31*m22^2-
2*C1r/deltasat^2*kv^2*kr*m32*m21*m22;
L44_43=-C1r/deltasat^2*kv^2*ky*m41*m22^2-
2*C1r/deltasat^2*kv^2*ky*m42*m21*m22;
L44_44=-C1r/deltasat^2*kv*kr^2*m21*m32^2-
2*C1r/deltasat^2*kv*kr^2*m22*m31*m32;
L44_45=-C1r/deltasat^2*kv*ky^2*m21*m42^2-
2*C1r/deltasat^2*kv*ky^2*m22*m41*m42;
L44_46=-C1r/deltasat^2*kr^2*ky*m41*m32^2-
2*C1r/deltasat^2*kr^2*ky*m42*m31*m32;
L44_47=-C1r/deltasat^2*kr*ky^2*m31*m42^2-
2*C1r/deltasat^2*kr*ky^2*m32*m41*m42;
L44_48=-2*C1r/deltasat^2*kpsi*kv*kr*m31*m22*m12;
L44_49 = -2*C1r/deltasat^2*kpsi*kv*kr*m32*m21*m12;
L44_50 =-2*C1r/deltasat^2*kpsi*kv*kr*m32*m22*m11;
L44_51 = -2*C1r/deltasat^2*kpsi*kv*ky*m41*m22*m12;
L44_52 = -2*C1r/deltasat^2*kpsi*kv*ky*m42*m21*m12;
L44_53 = -2*C1r/deltasat^2*kpsi*kv*ky*m42*m22*m11;
L44_54 =-2*C1r/deltasat^2*kpsi*kr*ky*m41*m32*m12;
L44_55 = -2*C1r/deltasat^2*kpsi*kr*ky*m42*m31*m12;
L44_56 = -2*C1r/deltasat^2*kpsi*kr*ky*m42*m32*m11;
L44_57 = -2*C1r/deltasat^2*kv*kr*ky*m41*m22*m32-
2*C1r/deltasat^2*kv*kr*ky*m42*m21*m32;
L44_58 =-2*C1r/deltasat^2*kv*kr*ky*m42*m22*m31;



L44   = L44_1+L44_2+L44_3+L44_4+L44_5+L44_6+L44_7+L44_8+L44_9+L44_10...

+L44_11+L44_12+L44_13+L44_14+L44_15+L44_16+L44_17+L44_18+L44_19+L44_20..

+L44_21+L44_22+L44_23+L44_24+L44_25+L44_26+L44_27+L44_28+L44_29+L44_30..

+L44_31+L44_32+L44_33+L44_34+L44_35+L44_36+L44_37+L44_38+L44_39+L44_40..

+L44_41+L44_42+L44_44+L44_45+L44_46+L44_47+L44_48+L44_49+L44_50+L44_51..

+L44_52+L44_53+L44_54+L44_55+L44_56+L44_57+L44_58;
```

```
L51=1/6*m11^3-1/6*m51^3-1/2*m61*m51^21/2*m21*m11^2;
L52=1/6*m12^3-1/6*m52^3-1/2*m62*m52^2+1/2*m22*m12^2;
L53=1/2*m11^2*m12-1/2*m51^2*m52-m61*m51*m52+m21*m11*m12-
1/2*m62*m51^2+1/2*m22*m11^2;
L54=1/2*m11*m12^2-1/2*m51*m52^2-1/2*m61*m52^2-
m62*m51*m52+1/2*m21*m12^2+m22*m11*m12;



%

R11 = N11*L11 + N12*L21 + N13*L31 + N14*L41 + N15*L51;
R13 = N11*L14 + N12*L24 + N13*L34 + N14*L44 + N15*L54;
R22 = N21*L13 + N22*L23 + N23*L33 + N24*L43 + N25*L53;
R24 = N21*L12 + N22*L22 + N23*L32 + N24*L42 + N25*L52;
%
Kcoef(index)=(1/8)*(3*R11+R13+R22+3*R24);
end
 end
  end
end
%
% Plot (L, crit TC) curve
%
plot(L_v,Kcoef,'b'),xlabel('L'),ylabel('{K}'),title('Constant T'),grid
%end
```

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LIST OF REFERENCES

1. Gokce, Mersin, Coupled *Stability Analysis of Close Proximity Ship Towing,* Thesis, Naval Postgraduate School, CA 2002

2. Jones, Garrett D., *Semi-rigid Towing Model for Analysis of Maneuvering in the Horizontal Plane*, Thesis, Naval Postgraduate School, CA 2001

3. Bernitsas M.M and Kekridis N.S., *Simulation and Stability of Ship Towing*, ISP, vol 32, no 369, 1985

4. Olcay Bulent, Effects of Observer Dynamics on Motion Stability of Autonomous Vehicles, Thesis, Naval Postgraduate School, CA 1994

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